Problem: Given a prime number $p$, prove that the polynomial congruence $(x + y)^n \equiv x^n + y^n \pmod{p}$ is true if and only if $n$ is a power of $p$.

Solution (by the Panel)

Let $P(x, y) = (x + y)^n - x^n - y^n = \sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k}$.

(a) If $n = p^a$, then all the coefficients of $P$ are divisible by $p$.

Proof: For $1 \leq j \leq p^a - 1$, $(\frac{p^a}{j}) = \frac{p^a - 1}{j - 1}$. If $j = rp^b$ where $(r,p) = 1$, then $\frac{p^a}{j} = \frac{p^{a-b}}{r}$ where $a - b \geq 1$ (since $j < p^a$). Thus $r$ must divide $(\frac{p^{a-1}}{j-1})$ (since $(\frac{p^a}{j})$ is an integer), and $(\frac{p^a}{j})$ is divisible by $p^{a-b}$.

(b) If $n$ is not a power of $p$ then not all $(\binom{n}{j})$ are divisible by $p$.

Proof: For $p^a < n < p^{a+1}$, let $c = n - p^a$ so $0 < c < p^a(p-1)$. Then $(\frac{n}{c}) = (\frac{p^a+c}{c}) = \prod_{j=1}^{c} \frac{p^a + j}{j}$. If $j = rp^b$ where $(r,p) = 1$ and $b < a$, then $\frac{p^a + j}{j} = \frac{p^{a-b} + r}{r}$. From this $(\frac{n}{c})$ equals a product of fractions none of whose numerators is a multiple of $p$.

Remark. Prof. Landy thought to have given a counter example to part (a). However, the assertion $f(x, y) = (x + y)^n - x^n - y^n \equiv 0$ is not meant as $f(x, y) \equiv 0$ for all integers $x, y$, but that every coefficient of the polynomial $f(x, y)$ is congruent to zero $(\pmod{p})$.

Also solved by:

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Six incorrect solutions were received.

Jason Anema (Jr. MA) submitted a late correct solution of Problem 5.