Problem of the Week
Solution of Problem No. 8 (Fall 2003 Series)

Problem: Suppose a pond contains \( x(t) \) fish at time \( t \) and \( x(t) \) changes according to \( \frac{dx}{dt} = x(1 - \frac{x}{x_0}) - f \), where \( x_0 \) is the equilibrium amount with no fishing and \( f > 0 \) is the constant rate of removal due to fishing. Assume \( x(0) = \frac{x_0}{2} \).

a) If \( f < \frac{x_0}{4} \), solve for \( x(t) \) and show that it tends to an equilibrium amount between \( \frac{x_0}{2} \) and \( x_0 \).

b) What happens if \( f \geq \frac{x_0}{4} \)?

Solution (by the Panel)

\[
\frac{dx}{dt} = x - \frac{x^2}{x_0} - f = -\frac{1}{x_0}(x - \frac{x_0}{2})^2 + \left( \frac{x_0}{4} - f \right).
\]

Let \( \frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}) = y(t) \), so that \( \sqrt{x_0} \, dy = dx \) and \( y(0) = 0 \).

Also let \( a^2 = \left| \frac{x_0}{4} - f \right| \). In these terms the D.E. is

\[
-dt = \frac{\sqrt{x_0} \, dy}{y^2 + a^2}
\]

where \( a = 0 \) if \( f = \frac{x_0}{4} \), negative if \( f < \frac{x_0}{4} \), and positive if \( f > \frac{x_0}{4} \).

(a) When \( f < \frac{x_0}{4} \),

\[
-t = \frac{\sqrt{x_0}}{2a} \log \frac{a-y}{a+y} + c.
\]

Since \( y(0) = 0 \), we have \( c = 0 \), and so

\[
e^{-\frac{2at}{\sqrt{x_0}}} = \frac{a-y}{a+y} = \frac{2a}{a+y} - 1, \text{ i.e. } y = \frac{2a}{1+e^{\frac{2at}{\sqrt{x_0}}}} - a.
\]

As \( t \to \infty \), clearly \( y \to a \), i.e. \( \frac{\sqrt{x_0}}{2a}(x - \frac{x_0}{2}) \to \sqrt{\frac{x_0}{4} - f} \), and so \( x \to \frac{x_0}{2} + \sqrt{\frac{x_0}{4} - fx_0} \).

(b) When \( f = \frac{x_0}{4} \), the original equation is \( \frac{dx}{dt} = -(x - \frac{x_0}{2})^2/x_0 \), which has the obvious constant solution \( x(t) = \frac{x_0}{2} = x(0) \). When \( f > \frac{x_0}{4} \),

\[
t = -\frac{\sqrt{x_0}}{a} \arctan \frac{y}{a} + c, \text{ where again } c = 0.
\]

Now

\[
-\tan \frac{x(t)}{\sqrt{x_0}} = \frac{y}{a} \text{ or } -\sqrt{f - \frac{x_0}{4}} \tan \frac{\sqrt{f - \frac{x_0}{4}}}{\sqrt{x_0}} \cdot t = \frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}), \text{ and so } x = \frac{x_0}{2} - \sqrt{fx_0 - \frac{x_0^2}{4}} \tan \frac{\sqrt{f - \frac{x_0}{4}}}{\sqrt{x_0}} \cdot t.
\]

This is a decreasing function of \( t \) which becomes 0 when \( \tan \frac{\sqrt{f}}{\sqrt{x_0} - \frac{x_0}{2}} = \frac{x_0}{2} \frac{1}{\sqrt{fx_0 - \frac{x_0^2}{4}}} \).

So the fish population becomes 0 in a finite time.

Solved by:

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Two incorrect solutions were received.