PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2003 Series)

Problem: Given $n$ points $P_0, P_1, \cdots, P_{n-1}$ ($n \geq 3$), equally spaced on the unit circle. Determine $\sum_{0 \leq k < \ell < n} |P_k P_\ell|^2$.

Solution (by Steven Landy, Fac. Physics at IUPUI); edited by the Panel.

\[
S = \sum_{(0 \leq k < \ell < n)} |P_k P_\ell|^2 \quad = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_k P_\ell|^2 \quad = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_0 P_{\ell-k}|^2
\]

(\text{Since } |P_0 P_{n-j}| = |P_0 P_j| \text{ and } |P_0 P_0| = 0.)

\[
S = \left(\frac{1}{2}\right) \sum_{(0 \leq \ell < n)} \sum_{(0 \leq j < n)} |P_0 P_j|^2 = \left(\frac{n}{2}\right) \sum_{(0 \leq \ell < n)} |P_0 P_\ell|^2.
\]

Since

\[
|P_0 P_\ell|^2 = (1 - e^{2\pi i \ell/n}) \cdot (1 - e^{-2\pi i \ell/n}) = 2 - e^{2\pi i \ell/n} - e^{-2\pi i \ell/n},
\]

\[
S = \left(\frac{n}{2}\right) \sum_{(0 \leq \ell < n)} \left(2 - e^{2\pi i \ell/n} - e^{-2\pi i \ell/n}\right) = n/2 \cdot 2n = n^2.
\]

Also solved by:

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