PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2005 Series)

Problem: Evaluate

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

Solution (by Georges Ghosn, Quebec)

The series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}$ converges because for $n \geq 1$,

$0 < \dfrac{n}{n^4+n^2+1} < \dfrac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

On the other hand, we have:

$$\frac{n}{n^4+n^2+1} = \frac{n}{(n^2+1)^2-n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1/2}{(n^2-n+1)} - \frac{1/2}{(n^2+n+1)}.$$

We pose $a_n = \dfrac{1/2}{(n^2-n+1)} \quad \Rightarrow \quad a_{n+1} = \dfrac{1/2}{(n^2+n+1)}$.

Therefore:

$$\begin{aligned} S_N &= \sum_{n=0}^{N} \frac{n}{n^4+n^2+1} = \sum_{n=0}^{N} (a_n - a_{n+1}) \\ &= (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_N - a_{N+1}) = a_0 - a_{N+1} \\ &= \frac{1}{2} - \frac{1/2}{N^2+N+1}. \end{aligned}$$

Finally:

$$\sum_{n=0}^{+\infty} \frac{n}{n^4+n^2+1} = \lim_{N\to\infty} S_N = \frac{1}{2}.$$

At least partially solved by:

<u>Undergraduates</u>: Iuri Bachnivski (Jr. ECE)

<u>Graduates</u>: Tom Engelsman (ECE), Miguel Hurtado (ECE), Eu Jin Toh (ECE), Qi Xu (Ch.E.)

Others: Alexander Bilik (Pomona, CA), Stephen Casey (Ireland), Mark Crawford (Sugar Grove, IL), Prithwijit De (Ireland), Nathan Faber (Cleveland, OH), Bob Hanek, Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Tim Lee (Junior, RPI), Aaditya Muthukumaran (Chennai, India), Rob Pratt (Raleigh, NC), Steve Spindler (Chicago)