PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2005 Series)

Problem: Let $f$ be a real–valued function with continuous non–negative derivative. Assume that $f(0) = 0$, $f(1) = 1$, and let $\ell$ be the length of the graph of $f$ on the interval $[0,1]$.

Prove that

$$\sqrt{2} \leq \ell < 2.$$

Solution (by the Panel)

The following inequalities are well known and easy to verify

$$\frac{\sqrt{2}}{2} (a + b) \leq \sqrt{a^2 + b^2} \leq a + b,$$

if $a \geq 0$, $b \geq 0$. The second one turns into equality if and only if $ab = 0$.

Now,

$$\ell = \int_0^1 \sqrt{1 + (f'(x))^2} \, dx.$$

Therefore, since $f'(x) \geq 0$,

$$\sqrt{2} = \frac{\sqrt{2}}{2} \int_0^1 (1 + f'(x)) \, dx \leq \int_0^1 (1 + f'(x)) \, dx = 2,$$

i.e.,

$$\sqrt{2} \leq \ell \leq 2.$$

If $\ell = 2$, then we must have

$$\sqrt{1 + (f'(x))^2} = 1 + f'(x), \quad \forall x \in [0,1].$$

This implies $f'(x) = 0$, $\forall x \in [0,1]$, thus $f = const$. The latter contradicts the conditions $f(0) = 0$, $f(1) = 1$.

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