**Problem of the Week**
Solution of Problem No. 6 (Fall 2005 Series)

**Problem:** Let \( \phi \) be the Euler function defined by \( \phi(1) = 1 \), and for any integer \( n > 1 \), \( \phi(n) \) is the number of positive integers \( \leq n \) and relatively prime to \( n \). Prove that for all real \( x \neq \pm 1 \),

\[
\sum_{m=0}^{\infty} (-1)^m \phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} = \frac{|x - x^3|}{(1 + x^2)^2}.
\]

**Solution** (by Georges Ghosn, Quebec)

We suppose first that \( |x| < 1 \), in this case the series \( \sum (-1)^m \phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} \) is absolutely convergent. Indeed, \( (-1)^m \phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} \leq (2m + 1)|x|^{2m+1} \) and it is easy to show that \( \sum (2m + 1)|x|^{2m+1} \) converges over \([-1, 1]\). On the other hand, we have for \( |x| < 1 \),

\[
\frac{x^{2m+1}}{1 + x^{4m+2}} = \frac{x^{2m+1}}{1 + (x^{2m+1})^2} = \sum_{n=0}^{+\infty} (-1)^n x^{(2m+1)(2n+1)} \quad (|x|^{2m+1} < 1).
\]

Therefore:

\[
\sum_{n=0}^{+\infty} (-1)^m \phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (-1)^{m+n} \phi(2m + 1)x^{(2m+1)(2n+1)}.
\]

In order to reorder terms of this double series, we must prove that it is absolutely convergent. Indeed this double series is absolutely convergent over any compact \([-\alpha, \alpha]\) \(0 \leq \alpha < 1\) because:

\[
|(-1)^m \phi(2m + 1)x^{(2m+1)(2n+1)}| \leq (2m + 1)\alpha^{(2m+1)(2n+1)}
\]

and

\[
\sum_{m=0}^{M} \sum_{n=0}^{+\infty} (2m + 1)\alpha^{(2m+1)(2n+1)} = \sum_{m=0}^{M} (2m + 1) \frac{\alpha^{2m+1}}{1 - \alpha^{4m+2}}
\]

and

\[
\forall m \geq 0 \quad (2m + 1) \frac{\alpha^{2m+1}}{1 - \alpha^{4m+2}} \leq (2m + 1) \frac{\alpha^{2m+1}}{1 - \alpha^2}
\]

and

\[
\sum (2m + 1)\alpha^{2m+1} \quad \text{converge for} \quad 0 \leq \alpha < 1.
\]
Therefore
\[ \sum_{m=0}^{+\infty} (-1)^m \Phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (-1)^{m+n} \Phi(2n+1) x^{(2m+1)(2n+1)} = \sum_{p=0}^{+\infty} a_p x^{2p+1} \]

where \( a_p \) is a summation extended over all terms \((-1)^{m+n} \Phi(2m+1)\) where \((2m+1)(2n+1) = 2p + 1\). But \((2m+1)(2n+1) = 2p + 1 \iff p = 2mn + m + n \Rightarrow (-1)^{m+n} = (-1)^p\).

Therefore \( a_p = (-1)^p \sum_{d|2p+1} \Phi(d) = (-1)^p (2p+1) \). (Euler Function Properties)

Finally
\[ \sum_{p=0}^{+\infty} (-1)^p (2p+1) x^{2p+1} = x \sum_{p=0}^{+\infty} (-1)^p x^{2p} + x^2 \sum_{p=1}^{+\infty} (-1)^p 2p x^{2p-1} \]
\[ = x \sum_{p=0}^{+\infty} (-1)^p (x^2)^p + x^2 \left( \sum_{p=0}^{+\infty} (-1)^p x^{2p} \right)' = \frac{x}{1 + x^2} + x^2 \left( \frac{1}{1 + x^2} \right)' \]
\[ = \frac{x}{1 + x^2} - \frac{2x^3}{(1 + x^2)^2} = \frac{x - x^3}{(1 + x^2)^2}. \]

Now for \(|x| > 1\), we have: \( \frac{x^{2m+1}}{1 + x^{4m+2}} = \frac{(\frac{1}{x})^{2m+1}}{1 + (\frac{1}{x})^{4m+2}} \) and \( \frac{1}{x} < 1 \). Therefore
\[ \sum_{m=0}^{+\infty} (-1)^m \Phi(2m + 1) \frac{x^{2m+1}}{1 + x^{4m+2}} = \sum_{m=0}^{+\infty} (-1)^m \Phi(2m + 1) \frac{(\frac{1}{x})^{2m+1}}{1 + (\frac{1}{x})^{4m+2}} \]
\[ = \frac{\frac{1}{x} - (\frac{1}{x})^3}{(1 + (\frac{1}{x})^2)^2} = \frac{x^3 - x}{(1 + x^2)^2}. \]

This completes the proof.

Also solved by:

**Undergraduates:** Arman Sabbaghi (Fr. Math & Stat)

**Graduates:** Eu Jin Toh (ECE)

**Others:** Prithwijit De (Ireland), Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff)