**Problem of the Week**
Solution of Problem No. 9 (Fall 2005 Series)

**Problem:** Triangle $T_1 = \triangle A_1B_1C_1$ is included in circle $K$. The perpendicular bisectors are drawn and extended through the interior of $T_1$ to their intersections $A_2, B_2, C_2$ with $K$. This process is repeated with the new triangle $T_2 = \triangle A_2B_2C_2$ to get new points $A_3, B_3, C_3$, etc.

Prove that
(a) the sequence $T_n$ has a subsequence that converges to some triangle $T_\infty$ and
(b) $T_\infty$ must be equilateral.

**Solution** (by the Panel)

First, (a) holds for any sequence of inscribed triangles $T_n = \triangle A_nB_nC_n$ by the following argument. Since $A_n$ belong to a compact set (the circle $K$), there is a convergent subsequence $A_{n_k} \to A_\infty \in K$. Apply the same argument to $B_{n_k}$ to get a convergent sub-subsequence $B_{n_{k_j}} \to B_\infty \in K$. Then, of course, $A_{n_{k_j}} \to A_\infty$. Finally, repeat this argument one more time to get a subsequence $C_{n_{k_{j_i}}} \to C_\infty \in K$. Then $T_{n_{k_{j_{i}}}}$ converges to $T_\infty = A_\infty B_\infty C_\infty$.

We need to show that in our case, any such $T_\infty$ will be equilateral. Let $\alpha_n, \beta_n, \gamma_n$, be the angles of $T_n$. It is easy to show that

$$\alpha_{n+1} = \frac{\pi}{2} - \frac{\alpha_n}{2}, \quad \beta_{n+1} = \frac{\pi}{2} - \frac{\beta_n}{2}, \quad \gamma_{n+1} = \frac{\pi}{2} - \frac{\gamma_n}{2}.$$ 

Those are recurrence equations with solutions

$$\alpha_n = \frac{-2\alpha_1}{(-2)^n} + \frac{2\pi}{3(-2)^n} + \frac{\pi}{3},$$

similarly for $\beta_n, \gamma_n$. Therefore, $\alpha_n \to \pi/3, \beta_n \to \pi/3, \gamma_n \to \pi/3$. Any subsequence has the same limit. Therefore, $T_\infty$ must be equilateral.

As George Ghosn pointed out, actually the whole sequence $T_n$ converges (to an equilateral triangle).

At least partially solved by:

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