**Problem of the Week**  
Solution of Problem No. 1 (Fall 2006 Series)

**Problem:** Let \( a > b > 0 \) be fixed numbers. Let \( Q \) be a convex planar quadrilateral with consecutive vertices \( A, B, C, D \) such that

\[
|AB| = |BC| = a, \quad |AD| = |DC| = b.
\]

Determine the extreme values of the distance between the center of mass of the vertices of \( Q \) and the center of mass of \( Q \) as a plane region.

**Solution** (by Georges Ghosn, Quebec; edited by the Panel)

![Diagram of the quadrilateral and its center of mass](image)

Observe that \( BD \) is the perpendicular bisector of \( AC \). Therefore in the coordinate system using \( BD \) as \( x \)-axis and \( AC \) as \( y \)-axis we have:

\[
A(0, -c) \quad B(-\sqrt{a^2 - c^2}, 0) \quad C(0, c) \quad D(\sqrt{b^2 - c^2}, 0), \quad 0 < c \leq b.
\]

The center of mass of the vertices of \( Q \) is:

\[
I \left( \frac{\sqrt{b^2 - c^2} - \sqrt{a^2 - c^2}}{4}, 0 \right).
\]

The center of mass of the plane region \( Q \) is:

\[
X_G = \frac{\int \int_Q x \, dx \, dy}{\text{area of } Q}, \quad Y_G = 0.
\]

But

\[
\int \int_Q x \, dx \, dy = \int_{-\sqrt{a^2 - c^2}}^{0} x \, dx \int_{-c}^{c} \left(1 + \frac{x}{\sqrt{a^2 - c^2}}\right) dy + \int_{0}^{\sqrt{b^2 - c^2}} x \, dx \int_{-c}^{c} \left(1 - \frac{x}{\sqrt{b^2 - c^2}}\right) dy
\]

\[
= -c(a^2 - c^2) + c(b^2 - c^2)
\]

and Area of \( Q \) is \( c\sqrt{a^2 - c^2} + c\sqrt{b^2 - c^2} \).

Therefore

\[
X_G = \frac{c(\sqrt{b^2 - c^2} + \sqrt{a^2 - c^2})(\sqrt{b^2 - c^2} - \sqrt{a^2 - c^2})}{3c(\sqrt{a^2 - c^2} + \sqrt{b^2 - c^2})} = \frac{\sqrt{b^2 - c^2} - \sqrt{a^2 - c^2}}{3}.
\]
Finally, the distance is \(|IG| = f(C) = \frac{\sqrt{(a^2 - c^2)} - \sqrt{(b^2 - c^2)}}{12}, 0 < c \leq b\). Next, \(f\) is an increasing continuous function on \([0, b]\) since

\[f'(C) = \frac{c}{12} \left( \frac{\sqrt{(a^2 - c^2)} - \sqrt{(b^2 - c^2)}}{(a^2 - c^2)(b^2 - c^2)} \right) > 0\]

on \((0, b)\). Therefore the extreme values are \(\frac{\sqrt{(a^2 - b^2)}}{12}\) (for \(c = b\)) and \(\frac{a - b}{12}\) (for \(c = 0\)). The last one is not reached since \(c \neq 0\).

Also, at least partially solved by:

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