PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2006 Series)

Problem:
Prove that the altitudes of a non-degenerate tetrahedron meet in a point if and only if each pair of opposite edges is orthogonal.

Solution (by Steven Landy, edited by the Panel)
Consider tetrahedron $ABCD$ with altitudes $Ae$ and $Df$. If $Ae$ and $Df$ intersect in $g$, then $ADefg$ are coplanar.

\[ Ae \perp \text{plane } (BCD) \Rightarrow Ae \perp BC \]
\[ Df \perp \text{plane } (ABC) \Rightarrow Df \perp BC. \]

Therefore, plane $(ADefg) \perp BC \Rightarrow AD \perp BC$. Similarly, $AB \perp DC, AC \perp BD$.

Now suppose $AD \perp BC$. Since $Ae \perp \text{plane } (BCD)$, $Ae \perp BC$. Thus, $BC \perp \text{plane } (ADe)$. Similarly, $BC \perp \text{plane } (ADF)$.
Therefore, $Ae$ and $Df$ are coplanar, so they intersect. Similarly all altitudes meet pairwise. Since no three of them are coplanar, they must meet in one point.

At least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE)

Others: Georges Ghosn (Quebec)