Problem:
Find all positive integers \( n \) such that the decimal expansion of \( n! \) ends with 2006 zeros but not with 2007 zeros. Use of computers or calculators is not allowed.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

The exponent of a prime \( p \) in the prime factorization of \( n! \) is equal to \( \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \) which is a finite summation because \( \left\lfloor \frac{n}{p^i} \right\rfloor = 0 \) for \( i > \log_p n \). Here \( [x] \) is the integer part of \( x \).

Therefore, \( n! = 2^a \cdot 5^b \cdots \) with \( a = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor \) and \( b = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor \).

But for \( n \geq 2 \), we have \( b < a \), therefore in order to have the decimal expansion of \( n! \) ends with exactly 2006 zeros, \( b \) must be equal to 2006.

But \( b = 2006 = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{n}{5^i} = \frac{n}{5} \sum_{i=0}^{\infty} \left( \frac{1}{5} \right)^i = \frac{n}{5} \times \frac{1}{1 - \frac{1}{5}} = \frac{n}{4} \), therefore \( n > 2006 \times 4 = 8024 \).

The exponent of 5 in 8025! is equal to \( \sum_{i=1}^{\infty} \left\lfloor \frac{8025}{5^i} \right\rfloor = 1605 + 321 + 64 + 12 + 2 = 2004 \).

Therefore, the smallest \( n \) for which the exponent of 5 in \( n! \) is equal to 2006, is 8035. Hence the integers are 8035, 8036, 8037, 8038 and 8039.

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