PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2006 Series)

Problem: Given $2n + 1$ positive integers with the property that if we remove any one of them, the remaining $2n$ numbers can be arranged in 2 sets of n numbers each with equal sums. Show that the $2n + 1$ numbers are equal.

Solution (by the Panel)

First it is easy to show that all numbers must have the same parity (all even, or all odd). Second it is also easy to show that if $\{x_i\}$ have the property, so do $\{ax_i + b\}$ with any two real a, b.

Let $x_{\min} = \min\{x_i\}_{i=1}^{2n+1}$. Then the set $\{y_i\}_{i=1}^{2n+1}$, where $y_i = x_i - x_{\min}$, also has the property and at least one of the y_i's, say y_1, is zero. Then all other y_i, $i = 2, \ldots, 2n + 1$ have to be even, too. We claim that $y_i = 0$, $\forall i$. If not, we can divide all y_i's by 2, and get a new set of integers with the property, and the first one will be zero, but not all will be zero. Then we can repeat this infinitely many tims, which is a contradiction, because each even positive integer can be divided only finitely many times by 2.

A solution provided by Prithwijit De shows that the statement is true even if x_i are not assumed to be integers. We will briefly sketch it. If $x = (x_1, \ldots, x_{2n+1})$, we know that

$$Ax = 0 \tag{1}$$

with some $(2n + 1) \times (2n + 1)$ matrix A so that

$$\begin{aligned} a_{ii} &= 0, \quad \forall\, i, \\ a_{ij} &= \pm 1 \quad , i \neq j, \end{aligned}$$

and for each i, exactly n a_{ij}'s are equal to $+1$. Then Rank $A \leq 2n$, because $(1, \ldots, 1)$ is an obvious solution, and we need to show that Rank $A = 2n$. Then the only solution to (1) will be a constant multiple of $(1, \ldots, 1)$. Let B be the $2n \times 2n$ matrix obtained by eliminating the last raw and the last column of A. Then $B^2 = I \pmod 2$, where I is the identity matrix, thus $\det B$ is odd, so B is invertible. Therefore, Rank $A = 2n$.

At least partially solved by:

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