Problem: Let $A_1, A_2, A_3, A_4$ be the areas of the faces of a tetrahedron. Let $\gamma_{ij}$ be the interior angle between the faces with areas $A_i$ and $A_j$. Prove that

$$A_4^2 = A_1^2 + A_2^2 + A_3^2 - 2A_1A_2 \cos \gamma_{12} - 2A_2A_3 \cos \gamma_{23} - 2A_3A_1 \cos \gamma_{31}. $$

Solution (by Steven Landy, edited by the Panel)

Let $\vec{A}_i$ be vectors perpendicular to the sides $A_i, i = 1, 2, 3, 4$, pointing to the exterior, with length equal to the area of the corresponding face $A_i$. Then it is easy to see that $\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0$ by representing each $\vec{A}_i$ as one half of the vector product of two edges.

Square both sides of

$$-\vec{A}_4 = \vec{A}_1 + \vec{A}_2 + \vec{A}_3$$

to get

$$A_4^2 = A_1^2 + A_2^2 + A_3^2 + 2A_1A_2 \cos(\pi - \gamma_{12})$$

$$+ 2A_1A_3 \cos(\pi - \gamma_{13}) + 2A_2A_3 \cos(\pi - \gamma_{23}),$$

which proves the equality.

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