Problem: Let \( P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \), where \( a_0, \ldots, a_n \) are integers. Show that if \( P \) takes the value 2007 for four distinct integral values of \( x \), then \( P \) cannot take the value 1990 for any integral value of \( x \). (Partial credit if you can prove it with “four” replaced by “five”.)

Solution (by Angel Plaza, ULPGC, Spain)

Let us consider the polynomial \( Q(x) = P(x) - 2007 \). Since \( P \) takes the value 2007 for four distinct integral values of \( x \), then \( Q \) has at least four different integral roots: \( r_1, \ldots, r_4 \).

\[
Q(x) = (x-r_1)(x-r_2)(x-r_3)(x-r_4)R(x),
\]
where \( R \) is also a polynomial with integral coefficients.

Let us suppose that there is an integer \( x^* \) such that \( P(x^*) = 1990 \), then \( Q(x^*) = -17 \), that is \( (x^*-r_1)(x^*-r_2)(x^*-r_3)(x^*-r_4)R(x^*) = -17 \). Since by hypothesis \( r_1, \ldots, r_4 \) are all different \( (x^*-r_1)(x^*-r_2)(x^*-r_3)(x^*-r_4) \) are four different divisors of \(-17\). But the only divisors of \(-17\) are 1, \(-1\), 17, \(-17\). Hence \( 1(-1)(17)(-17)R(x^*) = -17 \), which implies \( R(x^*) = \frac{1}{17} \). This contradicts the fact that \( R(x^*) \) is an integer.

Also solved by:

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A correct solution for the partial credit version was submitted by Subham Ghosh.