**Problem of the Week**

Solution of Problem No. 10 (Fall 2008 Series)

**Problem:** Find all differentiable functions \( f : [a, b] \rightarrow \mathbb{R} \) which have the property that

\[
\int_{\alpha}^{\beta} f(x) \, dx = \frac{f(\alpha) + f(\beta)}{2} (\beta - \alpha),
\]

(1)

whenever \( a \leq \alpha < \beta \leq b \).

**Solution** (by Brian Bradie, Christopher Newport University, VA)

Let \( f : [a, b] \rightarrow \mathbb{R} \) be differentiable and suppose \( f \) satisfies (1) whenever \( a \leq \alpha < \beta \leq b \). Differentiating (1) with respect to \( \beta \) yields

\[
f(\beta) = \frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2} f'(\beta)(\beta - \alpha),
\]

(2)

while differentiating (1) with respect to \( \alpha \) yields

\[-f(\alpha) = -\frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2} f'(\alpha)(\beta - \alpha).\]

(3)

If we subtract (3) from (2) we find

\[f(\alpha) + f(\beta) = f(\alpha) + f(\beta) + \frac{1}{2} (\beta - \alpha)(f'(\beta) - f'(\alpha)),\]

which simplifies to

\[f'(\beta) = f'(\alpha)\]

(4)

given that \( \alpha < \beta \). As (4) holds whenever \( a \leq \alpha < \beta \leq b \), it follows that \( f' \) is constant along \([a, b]\). Thus, if \( f : [a, b] \rightarrow \mathbb{R} \) is a differentiable function which satisfies (1) whenever \( a \leq \alpha < \beta \leq b \), then \( f \) is a linear function; that is, \( f(x) = mx + c \) for some constants \( m \) and \( c \).

Note that if we know \( f \) is at least twice continuously differentiable, then we may use the fact that the formula on the right-hand side of (1) is the trapezoidal rule, so

\[
\int_{\alpha}^{\beta} f(x) \, dx - \frac{f(\alpha) + f(\beta)}{2} (\beta - \alpha) = \frac{(\beta - \alpha)^3}{12} f''(\xi),
\]
where $\alpha < \xi < \beta$. Thus, (1) holds whenever $a \leq \alpha < \beta \leq b$ if and only if $f''(x) \equiv 0$; that is, $f(x) = mx + c$ for some constants $m$ and $c$.

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