PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2008 Series)

Problem: Suppose the interior of the unit circle is divided into two equal areas by an arc C (i.e., C is a non–self–intersecting path) with end points on the circle. Show that the length of C is at least 2.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let $\mathcal{C}$ be an arc with endpoints A and B on the unit circle, and assume that $\mathcal{C}$ divides the interior of the unit circle into two parts S_1 amd S_2 of equal areas. In what follows $\mathcal{C}_{PQ}$ will denote the part of the arc between the points P and Q and $|\mathcal{C}_{PQ}|$ the length of this part. Also $|PQ|$ will denote the length of the segment PQ. Clearly $|\mathcal{C}_{PQ}| \geq |PQ|$. If the center O of the unit circle belongs to $\mathcal{C}$ then:

$$|\mathcal{C}_{AB}| = |\mathcal{C}_{AO}| + |\mathcal{C}_{OB}| \geq |AO| + |OB| = 2.$$

Suppose that the center O of the unit circle does not belong to $\mathcal{C}$. Then one of the parts S_1 and S_2 does not contain O. Assume $O \notin S_1$. If the symmetric of $\mathcal{C}$ with respect to O does not intersect $\mathcal{C}$, then $\mathcal{C}$ and its symmetric divide the interior of the unit circle into three parts with disjoint interiors, one of which contains O and the other two are S_1 and its symmetric with respect to O. But this contradicts the fact that the area of S_1 (and hence of its symmetric) is half the area of the unit circle. Hence the symmetric of $\mathcal{C}$ with respect to O intersects $\mathcal{C}$. Therefore there exist on $\mathcal{C}$ two points D and E which are symmetric to each other with respect to O and such that the points A, D, E and B are placed on $\mathcal{C}$ in this order. Then:

$$|\mathcal{C}_{AB}| = |\mathcal{C}_{AD}| + |\mathcal{C}_{DE}| + |\mathcal{C}_{EB}| \geq |AD| + |DE| + |EB|.$$

On the other hand: $|DE| = |DO| + |OE|$, $|AD| + |DO| \geq |AO|$ and $|OE| + |EB| \geq |OB|$. Therefore:.

$$|AD| + |DE| + |EB| = (|AD| + |DO|) + (|OE| + |EB|) \geq |AO| + |OB| = 2.$$

Hence $|\mathcal{C}_{AB}| \geq 2$.

Also completely or partially solved by:

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