Problem: Show that

\[ n^k = \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k, \]

where

\( n \) and \( k \) are positive integers and \( n \geq k + 1 \).

Solution (by Elie Ghosn, Montreal, Quebec)

The Lagrange interpolating polynomial of degree \( \leq k \) (\( k \) positive integer) that passes through the \((k+1)\) points \((l, l^k)_{l=0,1,\ldots,k}\) is given by:

\[ p(x) = \sum_{l=1}^{k} \pi_l(x) l^k \quad \text{where} \quad \pi_l(x) = \prod_{\substack{j=0 \atop j \neq l}}^{k} \frac{(x-j)}{(l-j)} \]

\( p(x) \) is equal to \( Q(x) = x^k \) since both are of degree \( \leq k \) and \( p(l) = Q(l) \) for \( l = 0, \ldots, k \). Therefore for \( x = n \geq k + 1, n \) integer, we have:

\[ n^k = \sum_{l=1}^{k} \left( \prod_{\substack{j=0 \atop j \neq l}}^{k} \frac{(n-j)}{(l-j)} \right) l^k \]

but

\[ \prod_{\substack{j=0 \atop j \neq l}}^{k} (n-j) = \frac{\prod_{j=0}^{k} (n-j)}{n-l} = \frac{n!}{(n-k-1)!(n-l)} = \frac{n!}{(n-l)! \cdot (n-k-1)!} \cdot (n-l-1)! \]

and

\[ \prod_{\substack{j=0 \atop j \neq l}}^{k} (l-j) = \prod_{j=0}^{l-1} (l-j) \prod_{j=l+1}^{k} (l-j) = l! (-1)^{k-l}(k-l)!. \]
Therefore,

\[
n^k = \sum_{l=1}^{k} \frac{n!}{(n-l)!} \frac{(n-l-1)!}{(n-k-1)!} \frac{(-1)^{k-l}}{l!(k-l)!} t^l
= \sum_{l=1}^{k} (-1)^{k-l} \left( \binom{n}{l} \binom{n-1-l}{k-l} \right) t^l.
\]

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