PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2008 Series)

Problem: Find the minimum possible area of an ellipse which encloses a 3,4,5 right triangle.

Solution (by Sorin Rubinstein, Tel Aviv, Israel)

Let ABC be a $3,4,5$ right triangle. We are looking for an ellipse e which encloses the triangle ABC and such that the ratio $\dfrac{\text{Area}(e)}{\text{Area}(ABC)}$ is minimal. For every ellipse e which encloses the triangle ABC there exists an affine transformation that sends the ellipse e into the unit circle and the triangle ABC into a triangle $A'B'C'$ enclosed by the unit circle. Conversely, for every triangle $A'B'C'$ enclosed by the unit circle there exists an affine transformation that sends the triangle $A'B'C'$ into the triangle ABC and the unit circle into an ellipse which encloses the triangle ABC. Since affine transformations preserve the ratios of areas we must only find a triangle of maximal area enclosed by the unit circle. Moreover, since the area of $A'B'C'$ varies continuously when A', B' and C' move in the closed unit disk, $x^2 + y^2 \leq 1$, such a triangle exists. ('enclosed' is understood as placed inside, not necessarily inscribed) Let $A'B'C'$ be a triangle of maximal area enclosed by the unit circle. If one of the vertices, say A', is not placed on the circle than one can replace A' with the point A'' on which the altitude from A' intersects the unit circle and such that A' and $A^{''}$ are placed on the same side of $B'C'$ obtaining this way a triangle of which the area is strictly greater than the area of $A'B'C'$. This contradicts the maximality of the area of $A'B'C'$.

If follows that $A'B'C'$ is inscribed in the unit circle. Now, assume that $A'B'C'$, has two unequal sides, say $|A'B'| \neq |A'C'|$. Then one can replace A' with the point A'' on which the perpendicular bisector of the side $B'C'$ intersects the unit circle and such that A' and A'' are placed on the same side of $B'C'$. It follows that the altitude of the new triangle equals the distance between $B'C'$ and the tangent to the unit circle at A'' and is strictly greater than the altitude of $A'B'C'$ from A'. Therefore the triangle $A''B'C'$ has a strictly greater area than the triangle $A'B'C'$ which is a contradiction. It follows that the triangle $A'B'C'$ must be equilateral. Then the ratio between the area of the unit circle and the area of the triangle $A'B'C'$ is $\dfrac{\pi}{\frac{3}{4}\sqrt{3}} = \dfrac{4\pi}{3\sqrt{3}}$. Therefore the minimal area of an ellipse which encloses a $3,4,5$ triangle ABC is $\dfrac{4\pi}{3\sqrt{3}}\text{area}(ABC) = \dfrac{4\pi}{3\sqrt{3}} \cdot 6 = \dfrac{8\pi}{\sqrt{3}}$.

Also solved by:

<u>Graduates</u>: Huanyu Shao (CS)

<u>Others</u>: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Steven Landy (IUPUI Physics staff), Peyman Tavallali (Grad. student, NTU, Singapore), Bill Wolber Jr. (ITaP)

The Department has been without a fax machine since October 31. We expect fax service to be restored soon. Additional correct solutions are probably irretrievably stored in the old fax machine.