Problem: A set $F$ is called countable if either $F$ is finite or there is a one–to–one correspondence between the elements of $F$ and the natural numbers. Two sets $A$ and $B$ are called almost–disjoint if $A \cap B$ is finite.

Prove or disprove: There are uncountably many pairwise almost–disjoint sets of natural numbers (positive integers). In more formal language: Does there exist an uncountable set $F$ such that each element of $F$ is a set of natural numbers and each two elements of $F$ are almost–disjoint?

Solution (by Thierry Zell, Hickory, NC)

Let $I = (0.1, 1)$, and associate to each $x \in I$ the subset:

$$A_x = \{[10^n x] \mid n \in \mathbb{Z}, n \geq 1\}$$

Each subset $A_x$ is an infinite subset of the natural numbers. Through our choice of $I$, each set $A_x$ contains exactly one $n$–digit element for all $n \geq 1$, which represents the first $n$ decimals of $x$ in base 10. If $x$ and $y$ are two distinct elements of $I$, their decimal expansion must first differ at some rank $N$; we then have $|A_x \cap A_y| = N - 1$.

Thus, the collection of subsets $\{A_x \mid x \in I\}$ is an uncountable collection of pairwise almost–disjoint sets.

The problem was also solved by:

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