PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2009 Series)

Problem: Consider a rectangular array of dots with an even number of rows and an even number of columns. Suppose the dots are colored red or blue in such a way that every row has the same number of red and blue dots, and likewise every column. Whenever two dots of the same color are adjacent in a row or column, connect them with a line segment of that color. Show that the total number of blue segments must equal the total number of red segments.

Solution (by Clara Bennett, Undergrad, Purdue University)

Let us represent the array of red and blue dots with a $(2m) \times (2n)$ matrix, A. Each element A_{ij} which corresponds to a red dot has a value of $+\frac{1}{2}$, and each corresponding to a blue dot has a value of $-\frac{1}{2}$.
Each row and column has an equal number of blue and red dots. So, each row and column sums to zero. Let x be a $2n \times 1$ column vector of ones. Then, $\text{row}_i \cdot x = 0 \;\Rightarrow\; Ax = 0$.
Now, if we add two adjacent rows, the result will be $+1$ for each red segment and -1 for each blue. Let A_{vert} be the matrix of vertical line segments; i.e., $A_{vert} = VA$, wherre

$$V = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{a } (2m-1) \times (2m) \text{ matrix.}$$

The sum of all the elements of A_{vert} will give the difference in the number of vertical red and blue line segments.
$A_{vert}\, x$ gives a vector whose i'th component is the sum of the i'th row of A_{vert}. Since $A_{vert}\, x = (VA)x = V(Ax) = 0$, because $Ax = 0$, obviously $\sum_i (A_{vert}\, x)_i = 0$. A similar augument applies to the horizontal segments.

The problem was also solved by:

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