PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2009 Series)

Problem: Let us call a point (a, b) in the plane rational if both a and b are rational numbers. For a circle C, let $k(C)$ be the number of rational points on C. Prove that $k(C)$ must have one of the values $0, 1, 2, \infty$.

Solution (by Tairan Yuwen, Graduate student, Chemistry, Purdue University)

(1) The center of the circle (x_0, y_0) is a rational point. Suppose we can already find a rational point (x_1, y_1) on this circle, then we can use any rational number k as the slope of a line passing through (x_1, y_1). We can find two intersection points of the line and the circle (except the case of a tangent line); one of them is just (x_1, y_1), and the other one (x_2, y_2) can be solved using:

$$\begin{cases} \frac{y-y_1}{x-x_1} = k \\ (x-x_0)^2 + (y-y_0)^2 = R^2 = (x_1-x_0)^2 + (y_1-y_0)^2. \end{cases}$$

So we can get

$$\begin{cases} x_2 &= \frac{(k^2-1)(x_1-x_0)-2k(y_1-y_0)}{k^2+1} + x_0 \\ y_2 &= \frac{-2k(x_1-x_0)+(1-k^2)(y_1-y_0)}{k^2+1} + y_0. \end{cases}$$

Since x_0, y_0, x_1, y_1 and k are all rational numbers, (x_2, y_2) must be a rational point. So that means we can always find infinitely many rational points on the circle as long as we can find one rational point on this circle.

(2) The center of the circle (x_0, y_0) is not a rational point . In this case, suppose we can find at least 3 rational points A, B, C on the circle. Then the line joining A and B can be written as $l_1 : ax + by = c$, where a, b, c are all rational numbers, and it is also easy to show that the perpendicular bisector line of the line segment AB can be written as $l_1' : a'x + b'y = c'$; where a', b', c' are all rational numbers.

Similarly, the perpendicular bisector line of the line segment BC can also be written as: $l_2' : d'x + e'y = f'$, where d', e', f' are all rational numbers.

Obviously l_1' and l_2' intersect at (x_0, y_0), so we can get (x_0, y_0) by solving $\begin{cases} a'x + b'y = c' \\ d'x + e'y = f' \end{cases}$.

So (x_0, y_0) is a rational point, and that's contradictory to our assumption.

To sum up, we can only find $0, 1, 2, \infty$ rational points on any circle C in the plane, and here are examples for each of them:

$$
\begin{aligned}
&k(C) = 0: \quad x^2 + y^2 = \sqrt{3} \\
&k(C) = 1: \quad (x - \sqrt{3})^2 + y^2 = 3, \quad [(0,0)] \\
&k(C) = 2: \left(x - \frac{\sqrt{3}}{2}\right)^2 + y^2 = 1, \quad \left[\left(0, \frac{1}{2}\right) \text{ and } \left(0, -\frac{1}{2}\right)\right] \\
&k(C) = \infty: x^2 + y^2 = 1
\end{aligned}
$$

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Andy Bohn (Jr. Phys), Kilian Cooley (Fr.), Kun–Chieh Wang (Sr. Math), Brent Woodhouse (Fr. Science)

Graduates: Vitezslav Kala (Math)

Others: Mohamed Alimi (Tunisia), Andrea Altamura (Italy), Manuel Barbero (New York), Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Thierry Zell (Ph.D, Purdue 03)