**Problem of the Week**

Solution of Problem No. 10 (Fall 2010 Series)

**Problem:** Assume that the roots $r_1, r_2, r_3$ of the polynomial $p(x) = x^3 - 2x^2 + ax + b$ satisfy $0 < r_i < 1$, $i = 1, 2, 3$. Show that

(i) $2 \cdot \sqrt{1-r_1} \cdot \sqrt{1-r_j} \leq r_k$, $(i, j, k)$ a permutation of 1,2,3;

(ii) $8a + 9b \leq 8$;

(iii) the inequality in (ii) is best possible.

**Solution** (by Steven Landy, IUPUI Physics Dept. Staff)

(i) Since 2 is the sum of the roots, we have $r_3 = (1 - r_1) + (1 - r_2)$ where each bracket is positive. Then the arithmetic–geometric mean theorem says $r_3 \geq 2\sqrt{1-r_1\sqrt{1-r_2}}$ and likewise for the other permutations.

(ii) Multiplying the three inequalities from (i)

$$r_1 \geq 2\sqrt{1-r_3\sqrt{1-r_2}},$$
$$r_2 \geq 2\sqrt{1-r_1\sqrt{1-r_3}},$$
$$r_3 \geq 2\sqrt{1-r_1\sqrt{1-r_2}},$$

we get

$$r_1r_2r_3 \geq 8(1-r_1)(1-r_2)(1-r_3) = 8\left(1-(r_1+r_2+r_3)+(r_1r_2+r_2r_3+r_3r_1)-r_1r_2r_3\right).$$

Now using

$$(r_1 + r_2 + r_3) = 2 \quad (r_1r_2 + r_2r_3 + r_3r_1) = a \quad -r_1r_2r_3 = b$$

we get

$$-b \geq 8(1 - 2 + a + b) \quad \text{or} \quad 8a + 9b \leq 8.$$  

(iii) Using $r_1 = r_2 = r_3 = 2/3$ gives $8a + 9b = 8$. So the inequality is the best possible.

The problem was also solved by:
Undergraduates: Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student), Yixin Wang (So.), Lirong Yuan (Fr.)

Graduates: Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Steve Spindler (Chicago)