Problem: Show that for each real \( k \geq 3 \) the equation \((\ln x)^k = x\) for \( x \geq 1 \) has exactly two solutions \( r_k \) and \( s_k \) where \( r_k \to e \) and \( s_k \to \infty \), as \( k \to \infty \).

Solution: (by Kilian Cooley, Sophomore, Math & AAE)

Since there can be no solution at \( x = 1 \), we restrict our attention to \( x \) strictly greater than 1.

\[
(\ln x)^k = x, \quad x > 1
\]

\[
k(\ln \ln x) = \ln x, \quad x > 1
\]

\( \ln \ln x = 0 \) only when \( x = e \), but \((\ln e)^k = e\) is impossible for any real \( k \). Also, if \( 1 < x < e \), then \((\ln x)^k < 1 < x\), so any solution must occur at \( x > e \). Thus we can write

\[
k = \frac{\ln x}{\ln \ln x}, \quad k \geq 3, \quad x > e
\]

\[
x = \exp(\exp(u)), \quad u > 0
\]

\[
k = \frac{e^u}{u} = g(u)
\]

\[
\frac{d}{du}g(u) = e^u \frac{u - 1}{u^2}.
\]

From which we see that \( g(u) \) is monotonically increasing for \( u > 1 \) and monotonically decreasing for \( u < 1 \), has its only minimum at \( u = 1 \), and that \( \lim_{u \to 0} g(u) = \lim_{u \to \infty} g(u) = \infty \). Since \( g(u) \) is continuous it attains every real value \( k > g(1) = e < 3 \) exactly twice at \( v < 1 \) and \( w > 1 \). Also, it is clear that as \( k \to \infty \), \( v \) and \( w \) must tend to 0 and \( \infty \), respectively. Transforming backwards, this corresponds to \( x = r_k \) tending to \( \exp(\exp(0)) = e \) and to \( x = s_k \) tending to \( \infty \).

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