PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2010 Series)

Problem:d What is the smallest amount that may be invested at interest rate $i\%$, compounded annually, in order that we may withdraw $1^2, 2^2, 3^2, \ldots$ dollars at the end of the 1st, 2nd, 3rd, ... year, in perpetuity? (For $i = 10$, the answer is 2310 dollars.)

Solution (by Yue Pu, Freshman, Exchanged student from China)

Let $k = i\%$. At the nth year, we withdraw n^2 dollars which is $\dfrac{n^2}{1+k}$ dollars at the $(n-1)$th year, since $\dfrac{n^2}{1+k}$ dollars at the $(n-1)$th year, together with its interest at the nth year, is exactly n^2 dollars. We do this repeatedly and conclude that n^2 dollars's withdrawal is exactly $\dfrac{n^2}{(1+k)^n}$ dollars at first. Then, the smallest amount S is the following sum of infinite series

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(1+k)^n}$$

$$\begin{aligned}
\Rightarrow k^2 S =& ((1+k)^2 + 1 - 2(1+k))S \\
=& (1+k)^2 S + S - 2(1+k)S \\
=& (1+k) + 2^2 + \sum_{n=1}^{\infty} \frac{(n+2)^2}{(1+k)^n} + \sum_{n=1}^{\infty} \frac{n^2}{(1+k)^n} - 2 - 2\sum_{n=1}^{\infty} \frac{(n+1)^2}{(1+k)^n} \\
=& 1 + k + 4 - 2 + \sum_{n=1}^{\infty} \frac{(n+2)^2 + n^2 - 2(n+1)^2}{(1+k)^n} \\
=& k + 3 + \sum_{n=1}^{\infty} \frac{2}{(1+k)^n} \\
=& k + 1 + 2 \cdot \frac{k+1}{k} \\
\Rightarrow S =& (k+1)(k+2)/k^3 \quad \text{dollars}
\end{aligned}$$

when $k = i\% = 10\%$, the answer $S = 2310$ dollars.

The problem was also solved by:

<u>Undergraduates</u>: Ka Wang Chow (Sr. Science), Kilian Cooley (So.), David Elden (Sr. Mech. Engr), Eric Haengel (Jr. Math & Physics), Nathaniel Johnson (Fr. Math),

Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Michael Monte (Fr. Engr.), Jorge Ramos (So. Phys), Yixin Wang (So.)

<u>Graduates</u>: Rohit Jain (CS), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Huanyu Shao (CS), Ahmed Taher (EE), Tairan Yuwen (Chemistry)

<u>Others</u>: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Nathaniel Pattison (Sr. Christopher Newport Univ.), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Ph.D. student, Stanford Univ.), Dong-Gil Shin (12th grade, Newton North HS), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Américo Tavares (Queluz, Portugal), Mah Cheung Tsui (Jr. Stockdale HS, CA), Turkay Yolcu (Visiting at Purdue U.)