**Problem:** Let $A$ be a $4 \times 4$ matrix all of whose entries are $1$ or $-1$. List all possible values for the determinant of $A$. You must justify your answer without the use of a computer.

**Solution** (by Neacsu Adrian, Pitesti, Romania)

Denote by $D$ the determinant of the $4 \times 4$ matrix $(a_{ij})$. Adding line 4 to lines 1,2,3 will give the same value of $D$ and lines 1,2,3 will have elements from $\{-2, 0, 2\}$. Then extract factor 2 from each line 1,2,3 and $D$ will have factor 8. Therefore 8 divides $D$.

Absolute value of $D$ can be written: $|D| = |a_{11}d_1 + \ldots + a_{14}d_4| \leq |d_1| + \ldots + |d_4|$, where $d_i$ is the determinant of a $3 \times 3$ matrix $(b_{ij})$ having elements $\in \{-1, 1\}$. Using the same logic as above, determinant $d_i$ is divisible by 4 and $|d_i| = |b_{11}t_1 + b_{12}t_2 + b_{13}t_3| \leq |t_1| + |t_2| + |t_3|$, where $t_i$ is the determinant of a $2 \times 2$ matrix $(c_{ij})$ having elements $\in \{-1, 1\}$. But obviously $t_i \in \{-2, 0, 2\}$, $|t_i| \leq 2$. From here $|d_i| \leq 6$ and because $4|d_i$, we get $d_i \in \{-4, 0, 4\}$.

Finally $|D| \leq 16$.

2 examples of matrices for which $D = 8$ and $D = 16$ are: $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$

and $B = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

If 2 lines have the same elements [all 1 for example] then $D = 0$ and if we change the sign of all elements of 1 line from matrices $A$ and $B$ we get determinants with values $-8$ and $-16$.

We conclude $D \in \{-16, -8, 0, 8, 16\}$.

The problem was also solved by:

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