PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2010 Series)

Problem: Let A be a 4×4 matrix all of whose entries are 1 or -1. List all possible values for the determinant of A. You must justify your answer without the use of a computer.

Solution (by Neacsu Adrian, Pitesti, Romania)

Denote by D the determinant of the 4×4 matrix (a_{ij}). Adding line 4 to lines 1,2,3 will give the same value of D and lines 1,2,3 will have elements from $\{-2, 0, 2\}$. Then extract factor 2 from each line 1,2,3 and D will have factor 8. Therefore 8 divides D.

Absolute value of D can be written: $|D| = |a_{11}d_1 + \ldots + a_{11}d_4| \leq |d_1| + \ldots + |d_4|$, where d_i is the determinant of a 3×3 matrix (b_{ij}) having elements $\in \{-1, 1\}$. Using the same logic as above, determinant d_i is divisible by 4 and $|d_i| = |b_{11}t_1 + b_{12}t_2 + b_{13}t_3| \leq |t_1| + |t_2| + |t_3|$, where t_i is the determinant of a 2×2 matrix (c_{ij}) having elements $\in \{-1, 1\}$. But obviously $t_i \in \{-2, 0, 2\}$, $|t_i| \leq 2$. ¿From here $|d_i| \leq 6$ and because $4|d_i$, we get $d_i \in \{-4, 0, 4\}$.

Finally $|D| \leq 16$.

2 examples of matrices for which $D = 8$ and $D = 16$ are: $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

If 2 lines have the same elements [all 1 for example] then $D = 0$ and if we change the sign of all elements of 1 line from matrices A and B we get determinants with values -8 and -16.

We conclude $D \in \{-16, -8, 0, 8, 16\}$

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