Problem: Show that \(x^{400} + x^{380} + \cdots + x^{20} + 1\) is divisible by \(x^{20} + x^{19} + \cdots + x + 1\).

Solution: (by Kilian Cooley, Junior, Math & Aero Engineering, Purdue University)

Let \(f(x) = 1 + x + x^2 + \cdots + x^{20}\) and \(g(x) = 1 + x^{20} + x^{40} + \cdots + x^{400}\). \(f(x)\) divides \(g(x)\) if and only if every zero of \(f(x)\) is also a zero of \(g(x)\), including multiple zeros. \(f\) and \(g\) are both truncated geometric series, so they can be rewritten as

\[
f(x) = \frac{x^{21} - 1}{x - 1}, \quad g(x) = \frac{x^{420} - 1}{x^{20} - 1}
\]

From which one sees that the zeros of \(f\) and \(g\) are precisely those of \(x^{21} - 1\) and \(x^{420} - 1\) respectively, with the exception in both cases of \(x = 1\) where \(f(1) = g(1) = 21\). Therefore if \(f(r) = 0\), then \(r \neq 1\) and, noting that \(r^{20} \neq 1\),

\[
\begin{align*}
  r^{21} - 1 &= 0 \\
  r^{21} &= 1 \\
  (r^{21})^{20} &= r^{420} = 1^{20} = 1 \\
  r^{420} - 1 &= 0 \\
  g(r) &= 0
\end{align*}
\]

So any zero of \(f\) is also a zero of \(g\). Since the zeros of \(f\) are clearly the 21st roots of unity except 1, which are all distinct, every zero of \(f\) occurs exactly once in both \(f\) and \(g\). Therefore \(f\) divides \(g\). Q.E.D.

The problem was also solved by:

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