Problem: Let $P_i, 1 \leq i \leq n$, be $n$ distinct points in the plane, no three of which are in a straight line. Let $n \geq 3$. Prove that the shortest closed curve which goes through all of these points is a simple polygon, meaning a finite union of edges connected end to end, with each edge sharing each of its endpoints with exactly one other edge, and no other intersections of edges.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

There is a finite number $N \leq (n-1)!/2$ of closed curves $C_1, C_2, \ldots, C_N$. For a curve $C = (P_a, P_b, \ldots, P_z, P_a)$, let $l(C) = \|P_aP_b\| + \cdots + \|P_zP_a\|$ and without loss of generality assume that $l(C_1) \leq \cdots \leq l(C_N)$. We prove that the closed curve $C_1$ has no intersections. Suppose that $C_1$ has at last a intersection, $P_1P_3$ and $P_2P_4$ are edges and they intersect in the point $O$. We have $\|OP_1\| + \|OP_2\| > \|P_1P_2\|$, $\|OP_2\| + \|OP_3\| > \|P_2P_3\|$, $\|OP_3\| + \|OP_4\| > \|P_3P_4\|$ and $\|OP_1\| + \|OP_4\| > \|P_1P_4\|$. Therefore $\|P_1P_3\| + \|P_2P_4\| > \|P_2P_3\| + \|P_1P_4\|$ and $\|P_1P_3\| + \|P_2P_4\| > \|P_1P_2\| + \|P_3P_4\|$. Corresponding to each point $P_k$ there are exactly two entries, so we have the cases:

1. $C_1 = (P_1, P_3, \ldots, P_2, P_4, \ldots, P_1)$. Eliminate the edges $P_1P_3$ and $P_2P_4$, add the edges $P_1P_2$ and $P_3P_4$, so we obtain the closed curve $C'_1 = (P_1, P_2, \ldots, P_3P_4, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C'_1)$, a contradiction.

2. $C_1 = (P_1, P_3, \ldots, P_4, P_2, \ldots, P_1)$. Eliminate the edges $P_1P_3$ and $P_2P_4$, add the edges $P_2P_3$ and $P_1P_4$, so we obtain the closed curve $C'_1 = (P_1, P_4, \ldots, P_3P_2, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C'_1)$, a contradiction.

Hence the shortest closed curve has no intersections and it is a simple polygon.

Remark:

1. It is easy to see there can’t be closed proper subloops in $C_1$.
2. The easiest way to show there is some simple polygon through these points may be via the solution of this problem.
3. Some solutions similar to the one above were flawed because the “new” path might not be connected.

The problem was also solved by:

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