Problem: For every integer $n > 2$ let $L(n)$ denote the sum of the integers from 1 through $\lfloor n/2 \rfloor$ which are relatively prime to $n$, and let $U(n)$ denote the sum of the integers from $\lfloor n/2 \rfloor + 1$ through $n$ which are relatively prime to $n$. Prove that if $n$ is divisible by 4, then $U(n)/L(n) = 3$. ([ ] is the greatest integer function.)

**This problem was proposed by Steve Spindler. We also belatedly note that problem 10 of the Fall 2011 series was contributed by Hubert Desprez.**

Solution: (by Pierre Castelli, Math Teacher, Antibes, France)

Let $n > 2$ be an integer divisible by 4: $n = 4m$.

Let $j$ be a positive integer. Since $\gcd(j, n) = 1 \iff \gcd(j, 2m) = 1$ we can write:

$$L(n) = \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} j \quad \text{and} \quad U(n) = \sum_{2m+1 \leq j \leq 4m-1 \atop \gcd(j, 2m) = 1} j.$$  

For $1 \leq j \leq 2m - 1, \gcd(j, 2m) = 1 \iff \gcd(2m - j, 2m) = 1$, thus:

$$2L(n) = \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} j + \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} (2m - j) = \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} 2m.$$  

For $1 \leq j \leq 2m - 1, \gcd(j, 2m) = 1 \iff \gcd(2m + j, 2m) = 1$, so:

$$U(n) = \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} (2m + j) = \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} 2m + \sum_{1 \leq j \leq 2m-1 \atop \gcd(j, 2m) = 1} j = 2L(n) + L(n) = 3L(n).$$  

Finally:

$$\frac{U(n)}{L(n)} = 3.$$  

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