PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2012 Series)

Problem:
What is the maximum value of $a$ and the minimum value of $b$ for which 
$$
\left(1 + \frac{1}{n}\right)^{n+a} \leq e \leq \left(1 + \frac{1}{n}\right)^{n+b}
$$
for every positive integer $n$.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

The answer is $a_{\text{max}} = \frac{1}{\ln 2} - 1$ and $b_{\text{min}} = \frac{1}{2}$. Consider the functions $f, g, h : [1, \infty) \to \mathbb{R}$, 
$$
f(x) = \frac{1}{\ln(1 + 1/x)} - x \quad \text{with} \quad f(1) = \frac{1}{\ln 2} - 1 > 0.
$$
Applying L’Hospital twice we have

$$
\lim_{x \to \infty} f(x)^{L'Hospital} = \lim_{x \to \infty} \frac{\ln(1 + 1/x) + \frac{1}{x+1} L'Hospital}{\frac{1}{x+1} - \frac{1}{x}}
$$
\begin{align*}
&= \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}}{\frac{1}{x^2} - \frac{1}{(x+1)^2}} \\
&= \lim_{x \to \infty} \frac{\frac{1}{x+1} - \frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{x} + \frac{1}{x+1} = \frac{1}{2}.
\end{align*}
$$

Now we prove that $f$ is increasing. 
$$
f'(x) = \frac{g(x)}{(\ln(1 + 1/x))^2} \quad \text{where} \quad g(x) = \frac{1}{x} - \frac{1}{x+1} - (\ln(1 + 1/x))^2.
$$
\begin{align*}
g'(x) &= \left(1 - \frac{1}{x+1}\right)h(x) \quad \text{where} \quad h(x) = 2\ln(1 + 1/x) - \frac{1}{x+1} - \frac{1}{x} \quad \text{and} \\
h'(x) &= \left(\frac{1}{x+1} - \frac{1}{x}\right)^2 > 0.
\end{align*}

Therefore $h$ is increasing, $\lim_{x \to \infty} h(x) = 0$ and so $h < 0$. Therefore $g' < 0$, $g$ is decreasing, $\lim_{x \to \infty} g(x) = 0$ and so $g > 0$. Therefore $f' > 0$, and so $f$ is increasing. Hence $f(1) \leq f(x) < \frac{1}{2}$ so 
$$
\ln \left(1 + \frac{1}{x}\right)^{x+\frac{1}{x}+\frac{1}{x^2} - 1} \leq 1 < \ln \left(1 + \frac{1}{x}\right)^{x+\frac{1}{x}}
$$
and so for any $n \in \mathbb{N}^*$, 
$$
\left(1 + \frac{1}{n}\right)^{n+\frac{1}{n^2} - 1} \leq e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{n}}.
$$
Note that $b_{\text{min}} = \frac{1}{2}$ is optimal but there is no $n$ such that the equality holds in the right side of the double inequality.

The problem was also solved by:

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