Problem:
The numbers \( n_1, n_2, \ldots \) are generated one at a time as follows. If \( k = 1 \) or \( k = 2 \) or \( n_{k-1} \neq n_{k-2} \), roll an ordinary die once: \( n_k \) is the number rolled. If \( k > 2 \) and \( n_{k-1} = n_{k-2} \), roll the die perhaps repeatedly until you get a number different from \( n_{k-1} : n_k \) is this different number. Prove \( \lim_{k \to \infty} P(n_k = n_{k+1} = 6) \) exists and find the limit.

Solution: (by Talal AL Fares, High School Teacher, Hasbaya, Lebanon)

For \( k > 0 \) define the event \( A_k :"n_k = n_{k+1}"\) and let \( p_k = p(A_k) \), then
\[
p(n_k = n_{k+1} = 6) = \frac{p_k}{6}.
\]
We have
\[
p_{k+1} = p(A_{k+1} \cap A_k) + p(A_{k+1} \cap \overline{A_k}) = 0 + p(A_{k+1} / \overline{A_k}) p(\overline{A_k}) = \left( \frac{1}{6} \right)(1-p_k) = \frac{1-p_k}{6}.
\]
Clearly \( p_1 = \frac{1}{6} \), and by a simple induction it follows that \( p_k = \frac{1 - \left( \frac{1}{6} \right)^k}{7} \),
which does converge to \( \frac{1}{7} \).

Consequently, \( p(n_k = n_{k+1} = 6) \) converges to \( \frac{1}{42} \).

The problem was also solved by:

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