Solution of Problem No. 4 (Fall 2014 Series)

**Problem:**

**Show that if $f$ is a continuous function on the reals such that $f(t) > -t$ for all $t$, then if $g(x)$ is a solution of the initial value problem $\frac{dy}{dx} = f(y)$, $y(0) = 1$, there is no positive $x$ such that $g(x) = 0$.**

**Solution 1: (by Rustam Orazaliyev, Senior, Actuarial Science, Purdue University)**

$f(t) > -t, \quad \forall t \in R;$
$y'(x) = f(y(x)), \quad y(0) = 1$
Note then, $\quad y'(x) = f(y(x)) > -y(x) \quad \Rightarrow y'(x) + y(x) > 0.$
Let $F(x) = y(x)e^x$. Then $F(0) = 1$ and $F'(x) = e^x(y'(x) + y(x)) > 0$, $x > 0$, so by the mean value theorem $F(x) > F(0) = 1$, $x > 0$, and so $y(x) > e^{-x}$, $x > 0$.

**Solution 2: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)**

Suppose there exists certain positive $x$ that $g(x) = 0$ and let's assume the smallest positive value can be found is $x_0$. Since $g(x)$ is a solution of $\frac{dy}{dx} = f(y)$, it has $g'(x_0) = f(g(x_0)) = f(0) > 0$. Since $g(x_0) = 0$ and $g'(x_0) > 0$, we can find an interval $(x_0 - \varepsilon, x_0)$ such that $g(x) < 0$ for $x \in (x_0 - \varepsilon, x_0)(\varepsilon > 0)$. Thus $g$ takes on both positive and negative values in $[0, x_0)$ and since $g$ is continuous the intermediate value theorem guarantees there is a $y$ in $(0, x_0)$ such that $g(y) = 0$ which contradicts our assumption that $x_0$ was the smallest 0 of $g$.

**The problem was also solved by:**

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