PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2014 Series)

Problem:
A cube is inscribed in the unit sphere \(x^2 + y^2 + z^2 = 1\). Let \(A, B, C, D\) denote the vertices of one face of the cube. Let \(O\) denote the center of the sphere, and \(P\) denote a point on the sphere. Show that

\[
\cos^2(POA) + \cos^2(POB) + \cos^2(POC) + \cos^2(POD)
\]

is independent of \(P\).

Solution: (by Kuang-Ru Wu, Graduate Student, Mathematics, Purdue University)

Because angles are independent of coordinates, we may assume \(A, B, C, D\) are \((\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\).

And denote \(P\) by \((x_1, x_2, x_3)\).

From \(\overrightarrow{OP} \cdot \overrightarrow{OA} = |\overrightarrow{OP}||\overrightarrow{OA}|\cos(\angle POA)\), and the other similar equalities for \(B, C, D\), we have

\[
\cos^2(\angle POA) + \cos^2(\angle POB) + \cos^2(\angle POC) + \cos^2(\angle POD)
\]

\[
= \left(\frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3\right)^2 + \left(-\frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3\right)^2 + \left(\frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3\right)^2 + \left(-\frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3\right)^2
\]

\[
= \frac{4}{3}(x_1^2 + x_2^2 + x_3^2)
\]

\[
= \frac{4}{3}
\]

since \(P\) is on unit sphere. Therefore, the quantity is independent of \(P\).

The problem was also solved by:

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