PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2014 Series)

Problem:

A cube is inscribed in the unit sphere $x^2+y^2+z^2=1$. Let A, B, C, D denote the vertices of one face of the cube. Let O denote the center of the sphere, and P denote a point on the sphere. Show that

$$\cos^2(POA)+\cos^2(POB)+\cos^2(POC)+\cos^2(POD)$$

is independent of P.

<u>Solution : (by Kuang-Ru Wu, Graduate Student, Mathematics, Purdue University)</u>

Because angles are independent of coordinates, we may assume A, B, C, D are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. And denote P by (x_1, x_2, x_3).

From $\overrightarrow{OP}\cdot\overrightarrow{OA}=|\overrightarrow{OP}||\overrightarrow{OA}|\cos(\angle POA)$, and the other similar equalities for B, C, D, we have

$$\begin{aligned}
&\cos^2(\angle POA)+\cos^2(\angle POB)+\cos^2(\angle POC)+\cos^2(\angle POD)\\
=&\left(\frac{1}{\sqrt{3}}x_1+\frac{1}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3\right)^2+\left(\frac{-1}{\sqrt{3}}x_1+\frac{1}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3\right)^2+\\
&\left(\frac{1}{\sqrt{3}}x_1+\frac{-1}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3\right)^2+\left(\frac{-1}{\sqrt{3}}x_1+\frac{-1}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3\right)^2\\
&=\frac{4}{3}(x_1^2+x_2^2+x_3^2)\\
&=\frac{4}{3},
\end{aligned}$$

since P is on unit sphere. Therefore, the quantity is independent of P.

The problem was also solved by:

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