PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2014 Series)

**Problem:**

**A bent is the union of the interior and boundary of a simple quadrilateral such that an interior angle formed by two adjacent edges exceeds $180°$ and the interior angle formed by the other two edges exceeds $90°$. Prove that the union of the interior and boundary of an acute triangle can not be the union of a finite number of bents which have disjoint interiors.**

**Solution 1 : (by Craig Schroeder, Postdoc. UCLA)**

For reference, let's color the bents. Let the vertex with the largest angle and the adjacent edges be red. Let the vertex with the second largest angle and the adjacent edges be green. Let the two acute angles of the bent be blue.

A finite number of bents cannot cover an edge between two acute angles. Color this edge yellow for reference. To see why, lets place one of the bents against the yellow edge so that it covers one of its yellow endpoints. Since the angles at the ends of the yellow segment are acute, the vertex placed here must be blue. If one of the bent's red edges is placed over the yellow edge, the red vertex will be on the yellow edge, and the other red edge will extend outside the triangle beyond the yellow line since its angle is larger than $180°$. Thus, a green edge must be placed over the yellow edge. At the other end of the segment is the green vertex, which is over $90°$. Just beyond this green vertex remains a now shorter yellow segment with acute angles at each side. This process has no way of terminating, so no finite number of bents can suffice.

**Solution 2 : (by Sorin Rubinstein, TAU Faculty, Tel Aviv, Israel)**

Let $P$ be a (filled) convex polygon. We shall prove that $P$ cannot be the union of a finite set of (filled) simple concave quadrilaterals which have disjoint interiors.

Indeed, assume the contrary, and let $n$ be the number of involved quadrilaterals. Every simple concave quadrilateral has an interior angle larger than $180°$. The vertex of such an angle cannot lay on the boundary of $P$. Moreover, since the sum of the angles around a point is $360°$, the angles which exceeds $180°$ of two of the quadrilaterals cannot share their vertex. Hence, the vertices of the interior angles which exceed $180°$ of the involved quadrilaterals are placed at $n$ different interior points of $P$. On the other hand, the sum of the angles of involved quadrilaterals around each of these points is $360°$, and some of the quadrilaterals must also have vertices on the boundary of $P$.

It follows that the sum of the interior angles of the $n$ involved quadrilaterals exceeds $n \cdot 360°$, which is a contradiction.

This proves our assertion and a fortiori the assertion of the problem.

**The problem was also solved by:**

Undergraduates: Adam Kline (Fr. Chem/Phys), Sameer Manchanda (Fr. CS)

Graduates: Kuang-Ru Wu (Math), Tairan Yuwen (Chemistry)

Others: Hubert Desprez (Paris, France), Rick Shilling & Bruce Grayson (Orlando, FL), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Luciano Santos (Teacher, Portugal), Mehtaab Sawhney (HS Student, Commack HS, NY), David Stoner (HS Student, Aiken, S. Carolina), William Wu (Quantitative Engineering Design Inc.)