Problem: Evaluate $I = \int_R \frac{1}{(x^2 + 1)y} \, dxdy$ where $R$ is the region in the upper half plane between the two curves $2x^4 + y^4 + y = 2$, $x^4 + 8y^4 + y = 1$.

Solution (by Mike Hamburg, St. Joseph’s H.S.)

Let $f(x) = y \geq 0$ such that $x^4 + 8y^4 + y = 1$. There is only one $y \geq 0$ satisfying this because $8y^4 + y$ is monotone increasing for $y \geq 0$. Then $2x^4 + 16y^4 + 2y = 2$, so $2x^4 + (2y)^4 + (2y) = 2$, so $2f(x)$ is the upper limit. Also note that $f(x)$ is defined only for $|x| \leq 1$. Then

$$
\int_R \frac{dxdy}{(x^2 + 1)y} = \int_{-1}^{1} \left( \int_{f(x)}^{2f(x)} \frac{dy}{y} \right) \frac{dx}{1 + x^2} = \int_{-1}^{1} \left( \log y \bigg|_{f(x)}^{2f(x)} \right) \frac{dx}{1 + x^2}
$$

$$
= \int_{-1}^{1} (\log(2f(x)) - \log(f(x))) \frac{dx}{1 + x^2}.
$$

But

$$
\log 2f(x) - \log f(x) = \log \frac{2f(x)}{f(x)} = \log 2
$$

for all $x$, so this becomes

$$
\int_{-1}^{1} \frac{\log 2}{1 + x^2} \, dx = (\log 2) \tan^{-1} x \bigg|_{-1}^{1} = (\log 2) \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \frac{\pi \log 2}{2}.
$$

Also solved by:

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