Problem: Suppose $A, B$ are real $n \times n$ matrices with $A + B = I$ (identity) and $\text{rank}(A) + \text{rank}(B) = n$. Show that $A^2 = A$, $B^2 = B$, $AB = BA = 0$.

Solution (by Vikram Buddhi, Grad. MA, edited by the Panel)

Let $V$ be a vector space of dimension $n$, on which $A$ and $B$ act. Let $\text{rank}(A) = r$, so $\text{nullity}(A) = n - r$, $\text{nullity}(B) = r$. Let $x \in \text{kernel}(A) \cap \text{kernel}(B)$. Then $Ax = 0$ and $(I - A)x = 0$, $\therefore x = 0$. Hence $V = \text{ker}(A) \oplus \text{ker}(B)$. Let arbitrary $x \in V$ be decomposed: $x = x_1 + x_2$, $x_1 \in \text{ker}(A)$, $x_2 \in \text{ker}(B)$. Then $BAX = BAX_1 + BAX_2 = 0 + ABX_2 = 0$, $\therefore BA = 0$, likewise $AB = 0$. Also $A - A^2 = AB = 0$, $A = A^2$; $B - B^2 = BA = 0$, $B = B^2$.

Also solved by:

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Two incorrect solutions were received.