**Problem of the Week**
Solution of Problem No. 11 (Spring 2001 Series)

**Problem:** For \( n = 1, 2, \cdots \), set \( S_n = \sum_{k=0}^{3n} \binom{3n}{k} \), \( T_n = \sum_{k=0}^{n} \binom{3n}{3k} \).
Prove that \( |S_n - 3T_n| = 2 \).

**Solution** (by Steven Landy, Faculty, Physics at IUPUI)

Let \( w = e^{2\pi i/3} \), so \( w^2 = \overline{w} \) (conjugate), \( w^3 = 1, 1 + w + w^2 = 0 \). Then

\[
S_n = \sum_{k=0}^{n} \binom{3n}{k} 1^k = (1 + 1)^{3n}
\]

by the Binomial Theorem. Also

\[
U_n = \sum_{k=0}^{n} \binom{3n}{k} w^k = (1 + w)^{3n}
\]

and

\[
V_n = \sum_{k=0}^{n} \binom{3n}{k} w^{2k} = (1 + \overline{w})^{3n}.
\]

Because \( 1^k + w^k + w^{2k} = 0 \) unless \( k \) is a multiple of 3, when it is 3,

\[
S_n + U_n + V_n = 3T_n,
\]

and so

\[
3T_n - S_n = U_n + V_n = (1 + w)^{3n} + (1 + \overline{w})^{3n} = (-w^2)^{3n} + (-\overline{w}^2)^{3n}
\]

\[
= (-1)^n [e^{4\pi in} + e^{-4\pi in}] = 2(-1)^n,
\]

so \( |S_n - 3T_n| = 2 \).

Also solved by:

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