Problem of the Week
Solution of Problem No. 12 (Spring 2001 Series)

Problem: Let \( S_n \) be the sum of lengths of all the sides and all the diagonals of a regular \( n \)-gon inscribed in a unit circle. Evaluate \( S_n \). Find \( \lim_{n \to \infty} S_n/n \).

Solution (by Aditya S. Utturwar, Grad. AE, Georgia Inst. Tech., edited by the Panel)

One concludes from geometry that
\[
S_n = n \sum_{k=1}^{n-1} \sin k\theta, \quad \theta = \pi/n.
\]

Using the identity \( 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \), we have
\[
2 \sin \theta (\sin \theta + \sin 2\theta + \cdots + \sin (n-1)\theta)
= (\cos 0 - \cos 2\theta) + (\cos \theta - \cos 3\theta) + (\cos 2\theta - \cos 4\theta) + \cdots + (\cos (n-2)\theta - \cos n\theta)
= \cos 0 + \cos \theta - \cos (n-1)\theta - \cos n\theta
= 1 + \cos \theta + \cos \theta - (-1)
= 2(1 + \cos \theta).
\]

Hence
\[
\frac{S_n}{n} = \frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} = \cot(\theta/2), \quad \text{so that}
\]
\[
S_n = n \cot(\pi/2n) \quad \text{and}
\]
\[
\lim_{n \to \infty} \frac{S_n}{n} = \lim \cot(\pi/2n) = \infty.
\]

Remark. The Problem was supposed to ask for \( \lim S_n/n^2 \). By the above
\[
\lim_{n \to \infty} \frac{S_n}{n^2} = \lim \frac{\cos(\pi/2n)}{n \sin(\pi/2n)} = \lim \frac{1}{(\pi/2)(2n/\pi) \sin(\pi/2n)} = \frac{2}{\pi}
\]

Complete or partial solutions were received also from:
Undergraduates: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. EE/MA)
Faculty: Steven Landy (Phys. at IUPUI)