Problem: Let $p$ be a prime number and let $J$ be the set of all $2 \times 2$ matrices, \[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\] where $a, b, c, d \in \{0, 1, \ldots, p - 1\}$, and which satisfy $a + b \equiv 1 \pmod{p}$ and $ad - bc \equiv 0 \pmod{p}$. How many matrices are in $J$?

Solution (by Steven Landy, Fac. Phys. at IUPUI)

- $a$ can take on $p$ values: $0, 1, \ldots, p - 1$; $b \equiv 1 - a$ is then fixed.
  - If $a \equiv 0$ then $b \equiv 1, c \equiv 0$, while $d$ can be one of $0, 1, \ldots, p - 1$.
  - If $a \equiv 1$ then $b \equiv 0, d \equiv 0$, while $c$ can be one of $0, 1, \ldots, p - 1$.
  - If $a \not\equiv 0, a \not\equiv 1$, then $b \not\equiv 0$ and in $ad \equiv bc$, $d$ can be any of $0, 1, \ldots, p - 1$; and $c \equiv ad^{-1}$, where $b^{-1}$ is the unique reciprocal of $b \not\equiv 0 \pmod{p}$.

Thus, for any choice of $a$ there are $p$ ways to assign the remaining terms. Hence, the cardinality of $J$ is $p^2$.

Also solved by:

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