PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2002 Series)

Problem: Prove that, for odd positive integers n, $\binom{2n}{2} + \binom{2n}{6} + \binom{2n}{10} + \cdots = 2^{2n-2}$.

Solution (by many of the solvers, edited by the Panel)

$\binom{2n}{2+4j} = \binom{2n}{2n-2-4j}$.

We have since n is odd, $2n-2-4j$ is divisible by 4, say $2n-2-4j = 4k$. Then $\sum \binom{2n}{2+4j} = \sum \binom{2n}{4k}$, hence $2S = \sum \binom{2n}{2\ell}$. But $\sum \binom{2n}{2\ell} = \sum \binom{2n}{2\ell+1}$, hence $\sum \binom{2n}{2\ell} = \frac{1}{2} \sum \binom{2n}{j} = \frac{1}{2} 2^n$. Thus $S = \frac{1}{4} 2^n = 2^{n-2}$.

Also solved by:

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Others: J.L.C. (Fishers, IN), Prithwijit De (STAT at U. Coll. Cork, Eire), John G. DelGreco (MA, Loyola U.), Rob Pratt (Gr. U.N.C., Chapel Hill, NC), Leo Sheck (Medical Sch, U. Auckland, NZ), Aditya Utturwar (Grad. AE, Georgia Tech)

Three unacceptable solutions were received.

The solution of Problem 8 presented recently was inadequate because incomplete and needs to be supplemented. As in this presented solution the given 5 points are $O(0,0,0)$, $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, $P(x,y,z)$. The point P which makes the volumes of $POAB, POBC, POCA$ equal must be at equal distance d, from the planes $x=0$, $y=0$, $z=0$ because the areas of OAB, OBC, and OCA equal $\frac{1}{2}$. The common volume must be $\frac{1}{6}$ since $|OABC| = \frac{1}{6}$. Hence d, the distance must be 1, $\frac{1}{6}$ being $\frac{1}{3} \times \frac{1}{2} \times 1$. The only possible points P are $(\pm1, \pm1, \pm1)$. Because of symmetry it suffices to consider $P_1(1,1,1), P_2(1,1,-1), P_3(1,-1,-1), P_4(-1,-1,-1)$. For each of these we show that $|P_iABC|$ is not equal to $|OABC|$ by showing that the distance of P_i from the plane S of ABC is not the same as the distance of O from S. The distance of a point $P(x_0, y_0, z_0)$ from a plane $ax+by+cz+d=0$ is given by $|(ax_0+by_0+cz_0+d)/\sqrt{a^2+b^2+c^2}|$. The plane S is given by $x+y+z-1=0$, hence $\text{dist}(OS)$ is $1/\sqrt{3}$, $\text{dist}(P_1S)$ is $2/\sqrt{3}$, $\text{dist}(P_2S) = 0$, $\text{dist}(P_3S) = 2/\sqrt{3}$ and $\text{dist}(P_4S)$ is $4/\sqrt{3}$. Since no P_i has the same distance from S as O, the equality of volumes is not possible.