**Problem of the Week**  
Solution of Problem No. 12 (Spring 2002 Series)

**Problem:** Determine all the real $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $A^2 + A + I = 0$.

**Solution** (by Damir Dzhafarov, Fr. MA, edited by the Panel)

Expanding the equation yields the system

\[
\begin{cases}
  a^2 + a + bc + 1 = 0 & \text{(i)} \\
  ab + bd + b = 0 & \text{(ii)} \\
  ac + cd + c = 0 & \text{(iii)} \\
  d^2 + d + bc + 1 = 0 & \text{(iv)}
\end{cases}
\]

Solving (i) and (iv) for $a$ and $d$ respectively gives

\[a = \frac{-1 \pm \sqrt{-3-4bc}}{2}, \quad d = \frac{-1 \pm \sqrt{-3-4bc}}{2},\]

whence it follows that $bc \leq -3/4$ and consequently $b \neq 0$ and $c \neq 0$. Thus, (ii) may be divided by $b$ and (iii) by $c$ to obtain $a + d = -1$, which is satisfied only if $bc \leq -3/4$. The sought solutions are therefore

\[A = \begin{pmatrix}
\frac{-1 \pm \sqrt{-3-4mn}}{2} & m \\
-1 \pm \sqrt{-3-4mn} & \frac{n}{2}
\end{pmatrix},\]

where $m, n$ arbitrary real numbers except $mn < -3/4$.

Also solved by:

**Undergraduates:** Eric Tkaczyk (Jr. EE/MA)

**Graduates:** Dharmashankar Subramanian (ChE)

**Faculty:** Steven Landy (Phys. at IUPUI)

**Others:** Dane Brooke (Boeing, Seattle), J.L.C. (Fishers, IN), John G. DelGreco (MA, Loyola U.), Leo Sheck (Medical Sch, U. Auckland, NZ),

Five unacceptable solutions were received.