PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2002 Series)

**Problem:** If $P,Q,R,S$ are polynomials, show that $\int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$ is divisible by $(x - 1)^4$.

**Solution** (by Chris Lomont, graduate (MA), edited by the Panel)

Let $F(x) = \int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$. Clearly $F(x)$ is a polynomial in $x$. $F(1) = 0$ (clearly: $\int_1^1 \cdots = 0$), so $(x - 1)$ divides $F(x)$.

$$F'(x) = (RS) \int_1^x PQ + (PQ) \int_1^x RS - (QR) \int_1^x PS - (PS) \int_1^x QR,$$

so $F'(1) = 0$, thus $(x - 1)^2 | F(x)$.

$$F''(x) = (RS)' \int_1^x PQ + RS PQ + (PQ)' \int_1^x RS + PQ RS - (QR)' \int_1^x PS - QR PS - (PS)' \int_1^x QR - PS QR = (RS)' \int_1^x PQ + (PQ)' \int_1^x RS - (QR)' \int_1^x PS - (PS)' \int_1^x QR,$$

so $F''(1) = 0$, and $(x - 1)^3 | F(x)$.

$$F'''(x) = (RS)'' \int_1^x PQ + (RS)' PQ + (PQ)'' \int_1^x RS + (PQ)' RS - (QR)'' \int_1^x PS - (QR)' PS - (PS)'' \int_1^x QR - (PS)' QR = (RS)'' \int_1^x PQ + (PQ)'' \int_1^x RS - (QR)'' \int_1^x PS - (PS)'' \int_1^x QR + (RS)' (PQ) + (PQ)' RS - [(QR)' (PS) + (PS)' QR].$$

The terms without integral factors are $(PQRS)' - (PQRS)' = 0$ so $F'''(1) = 0$, and $(x - 1)^4$ divides $F(x)$. 
Note $P = y, Q = y, R = 3, S = 4$ gives

\[
F(x) = \int_1^x y^2 \int_1^x 12 - \int_1^x 3y \int_1^x 4y
\]

\[
= 12\left(\frac{y^3}{3} \bigg|_1^x \cdot y \bigg|_1^x\right) - 12\left(\frac{y^2}{2} \bigg|_1^x \cdot \frac{y^2}{2} \bigg|_1^x\right)
\]

\[
= 4(x^3 - 1)(x - 1) - 3(x^2 - 1)(x^2 - 1)
= (x - 1)^2[4(x^2 + x + 1) - 3(x + 1)^2]
= (x - 1)^2[4x^2 + 4x + 4 - 3x^2 - 6x - 3]
= (x - 1)^2[x^2 - 2x + 1]
= (x - 1)^4
\]

so $(x - 1)^5 \not\in F(x)$ in general.

Also solved by:

**Graduates:** Tom Hunter (MA)

**Faculty:** Steven Landy (Physics at IUPUI)

**Others:** J.L.C. (Fishers, IN)

One unacceptable solution was received.