PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2003 Series)

Problem: Given $b_1 = 1$, $b_n = 1 - \frac{1}{n^2} \sum_{k=1}^{n-1} k^2 b_k$ for $n \geq 2$, sum the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. You may use that $\sum_{n=1}^{\infty} (-1)^{n-1}/n^2 = \pi^2/12$.

Solution (by Namig Mammadov, Baku, Azerbaijan)

$$\sum_{k=1}^{n} k^2 b_k = \sum_{k=1}^{n-1} k^2 b_k + n^2 b_n = n^2 - n^2 b_n + n^2 b_n = n^2,$$

so we get

$$b_n = 1 - \frac{1}{n^2} \cdot (n-1)^2 = \frac{2n - 1}{n^2} = \frac{2}{n} - \frac{1}{n^2}.$$  

Hence

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{2}{n} - \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 2 \ln 2 - \frac{\pi^2}{12}.$$  

Also solved by:

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There was one unacceptable solution.