PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2004 Series)

Problem: Let D_n be the region below the hyperbola $y = 1/x$ for $1 \le x \le n$ and above the union of the rectangles with base $k \le x \le k+1$ and height $2/(2k+3)$ for $k = 1, \cdots, n-1$. Determine $\lim_{n\to\infty}$ (area of D_n).

Solution (by the Panel)

$$D_n = \log n - 2\sum_{k=1}^{n-1} \frac{1}{2k+3},$$

where
$$\begin{aligned} 2\sum_{k=1}^{n-1} \frac{1}{2k+3} &= \sum_{k=1}^{n} \frac{1}{k} - (2\sum_{k=1}^{n} \frac{1}{2k} - 2\sum_{k=1}^{n} \frac{1}{2k+1}) - \frac{2}{3} \\ &= \sum_{k=1}^{n} \frac{1}{k} + 2\sum_{k=1}^{2n+1} (-1)^{k-1}\frac{1}{k} - \frac{8}{3}. \end{aligned}$$

Hence
$$\begin{aligned} \lim_{n\to\infty} D_n &= \lim(\log n - \sum_{k=1}^{n} \frac{1}{k}) - 2\lim(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n+1}) + \frac{8}{3} \\ &= \gamma + \frac{8}{3} - 2\log 2, \end{aligned}$$

where γ is Euler's constant.

Also solved by:

<u>Undergraduates</u>: Adam Welborn (So. CS)

<u>Graduates</u>: Jianguang Guo (Phys)

<u>Faculty</u>: Steven Landy (Phys, IUPUI)

<u>Others</u>: Georges Ghosn (Quebec), Jonathan Landy (Cal. Tech.)

Three unacceptable solutions were received.

We received late solutions of Problem 2 from: Jignesh Vidyut Mehta (Jr. Phys) and Sandeep Nandy (So. Eng.)