Problem:
(a) Show that the number of ways in which an odd positive integer \( n \) can be written as a sum of two or more consecutive positive integers is equal to the number of divisors \( d \) of \( 2n \) such that \( 1 < d < \sqrt{2n} \).

(b) Give all these sums for \( n = 45 \).

Solution (by Georges Ghosn, Quebec, edited by the Panel)

Notice first, that each such sum is uniquely determined by the number \( d \) of its terms.

(a) (i) Let’s show first that if \( n \) can be written as a sum of \( d \) (\( d > 1 \)) consecutive positive integers then \( d \) is divisor of \( 2n \) and \( 1 < d < \sqrt{2n} \).

Indeed \( n = m + (m + 1) + \cdots + (m + d - 1) \) where \( m > 0 \)

\[
\Leftrightarrow n = md + \frac{d(d-1)}{2} \Leftrightarrow 2n = d(2m + d - 1) \Rightarrow d \text{ is a divisor of } 2n
\]

but

\[
2m - 1 > 0 \Rightarrow d^2 < d(2m + d - 1) = 2n
\]

\[
\Rightarrow d < \sqrt{2n}
\]

(ii) Let’s show now that if \( d \) is a divisor of \( 2n \) and \( 1 < d < \sqrt{2n} \) then there is a unique \( m > 0 \) so that \( n = m + (m + 1) + \cdots + (m + d - 1) \).

Indeed \( 2n = d \times k \) but \( d < \sqrt{2n} \Rightarrow k > \sqrt{2n} > d \)

\[
2n = d \times k = d(2m + d - 1) \Rightarrow 2m = k - (d - 1)
\]

but \( n \) is odd \( \Rightarrow d \) and \( k \) don’t have the same parity

\Rightarrow (d - 1) \text{ and } k \text{ have the same parity}

\Rightarrow m = \frac{k-(d-1)}{2} > 0 \text{ is an integer, and is unique.}

Consequently the 2 sets have the same number of elements.

(b) \( n = 45 \) \( \text{ the divisors of } 2n = 90 \text{ are } d \text{ and } 1 < d \leq 9 \) \( \text{ are } d = 2, 3, 5, 6, 9. \)

\[
45 = 22 + 23 \quad (m = 22, \quad d = 2)
45 = 14 + 15 + 16 \quad (m = 14, \quad d = 3)
45 = 7 + 8 + 9 + 10 + 11 \quad (m = 7, \quad d = 5)
45 = 5 + 6 + 7 + 8 + 9 + 10 \quad (m = 5, \quad d = 6)
45 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \quad (m = 1, \quad d = 9)
\]
Also solved by:

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