Problem of the Week
Solution of Problem No. 5 (Spring 2005 Series)

Problem: Suppose \( \{a_n\}_{n=1}^{\infty} \) be recursively defined by \( a_0 > 1, \ a_1 > 0, \ a_2 > 0, \)
\[
a_{n+3} = \frac{1 + a_{n+1} + a_{n+2}}{a_n}, \quad \text{for } n = 0, 1, 2, \ldots
\]
Show that \( a_n \) has period 8, i.e.
\[
a_{n+8} = a_n \quad \text{for any } n \geq 0.
\]

Solution (by the Panel)

Subtract the equations
\[
a_{n+3} a_n = 1 + a_{n+1} + a_{n+2}
\]
\[
a_{n+2} a_{n-1} = 1 + a_n + a_{n+1}
\]

to get
\[
a_{n+3} a_n - a_{n+2} a_{n-1} = a_{n+2} - a_n,
\]

or
\[
a_n (a_{n+3} + 1) = a_{n+2} (a_{n-1} + 1).
\]

Add \( a_n a_{n+2} \) to both sides to get
\[
a_n (1 + a_{n+2} + a_{n+3}) = a_{n+2} (1 + a_{n-1} + a_n).
\]

Using the recursive equation, we get
\[
a_n a_{n+1} a_{n+4} = a_{n+2} a_{n+1} a_{n-2}.
\]
Since all terms are positive, we cancel \( a_{n+1} \) to obtain
\[
a_{n-2} a_{n+2} = a_n a_{n+4}.
\]
Replace \( n \) by \( n - 2 \) to get
\[
a_n a_{n-4} = a_{n+2} a_{n-2}.
\]
The last two identities imply \(a_n a_{n-4} = a_n a_{n+4}\), therefore

\[a_{n-4} = a_{n+4}, \quad n \geq 4 \implies a_n = a_{n+8}, \quad n \geq 0.\]

Also solved by:

**Undergraduates:** Alan Bernstein, Justin Woo (Jr. ECE)

**Graduates:** Tom Engelsman, Niru Kumari (ME)

**Others:** Georges Ghosn (Quebec), Byungsoo Kim (Seoul Natl. Univ.), Jeff Ledford (Gainesville, GA), A. Plaza (ULPGC, Spain), Steve Spindler, Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

**Update on Problem No. 4:**
That problem was solved also by Justin Woo (Jr. ECE).