Problem: Let $\varepsilon$ be an ellipse, which is not a circle, with center $O$. Let $P \in \varepsilon$ be a point at which the angle between the tangent to $\varepsilon$ at $P$ and $\overrightarrow{OP}$ is minimal. Find the angle that $\overrightarrow{OP}$ makes with the major axis.

Solution (by Georges Ghosn, Quebec; edited by the Panel)

The ellipse equation is $\varepsilon : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and by symmetry we can assume that $P(x_1, y_1) \in \varepsilon$ is in the first quadrant. The equation of the tangent to $\varepsilon$ at $P$ is $\frac{Xx_1}{a^2} + \frac{Yy_1}{b^2} = 1$. Therefore the slope of the tangent is $m_1 = -\frac{b^2x_1}{a^2y_1}$ and the slope of $\overrightarrow{OP}$ is $m_2 = \frac{y_1}{x_1}$. Therefore the tangent of the angle between the above lines is

$$\frac{m_2 - m_1}{1 + m_2m_1} = \frac{a^2y_1^2 + b^2x_1^2}{(a^2 - b^2)x_1y_1} = \frac{a^2b^2}{(a^2 - b^2)x_1y_1}.$$ 

This angle is minimal if and only if this tangent is minimal and therefore $x_1y_1$ is maximal. But $\frac{x_1}{a} \cdot \frac{y_1}{b} \leq \frac{1}{2} \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \right) = \frac{1}{2} \Rightarrow x_1y_1 \leq \frac{ab}{2}$. Therefore $x_1y_1$ is maximal iff $\frac{x_1}{a} = \frac{y_1}{b} = \frac{\sqrt{2}}{2}$. Finally the angle that $\overrightarrow{OP}$ makes with the major axis is $\tan^{-1} \frac{y_1}{x_1} = \tan^{-1} \left( \frac{b}{a} \right)$.

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