PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2006 Series)

Problem: Let $a_1 = \sqrt{2}$, $a_2 = (\sqrt{2})^{a_1}$, and

$$a_n = (\sqrt{2})^{a_{n-1}}, \quad n = 2, 3, \ldots$$

Show that $\lim_{n \to \infty} a_n$ exists and determine its value.

Solution (by Tomek Czajka, CS graduate student; edited by the Panel)

Let $a_0 = 1$. Then $a_1 = \sqrt{2}^{a_0}$. If $a_n < 2$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^2 = 2$. Since $a_0 < 2$, by induction, $a_n < 2$ for all $n$. If $a_n < a_{n+1}$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^{a_{n+1}} = a_{n+2}$. Since $a_0 = 1 < \sqrt{2} = a_1$, by induction we get that the sequence $(a_n)$ increases. Since $(a_n)$ increases, starts from 1 and is bounded by 2, it has a limit between 1 and 2 (inclusive).

Let $a = \lim_{n \to \infty} a_n$. We have $1 \leq a \leq 2$. Also:

$$a = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2}^{a_n} = \sqrt{2}^{\lim_{n \to \infty} a_n} = \sqrt{2}^a$$

$$a^2 = 2^a$$

$$2 \ln a = a \ln 2$$

$$2 \ln a - a \ln 2 = 0$$

Let $f(x) = 2 \ln x - x \ln 2$ for $x > 0$. Then $f(a) = 0$ and $f(2) = 0$. For $1 \leq x \leq 2$, we have $f'(x) = 2/x - \ln 2 > 2/2 - \ln e = 0$. Therefore $f$ increases on the interval $[1,2]$ and thus is injective on this interval. Since $a \in [1, 2]$ and $f(a) = f(2)$, we must have $a = 2$.

Answer:

$$\lim_{n \to \infty} a_n = 2.$$