**Problem of the Week**  
Solution of Problem No. 6 (Spring 2006 Series)

**Problem:** Let $C$ be the curve whose equation is 

$$y = P(x),$$

where $P(x)$ is a real polynomial having at least one real zero $x_0 \neq 0$.

Prove that there exists a point on the curve $C$, different from $(x_0, 0)$, whose distance to $(0, 0)$ is $|x_0|$.

**Solution** (by the Panel)

Intuitively, the statement is clear. Let $K$ be the circle with center $(0, 0)$ and radius $|x_0|$. Then $K$ and $C$ have at least one common point $(x_0, 0)$ and they are not tangent to each other at this point (because $K$ has a vertical tangent, $C$ has a finite slope $P'(x_0)$). Therefore, $C$ enters $K$, and it has to leave somewhere because $P(x)$ is defined everywhere.

To make those arguments precise, introduce the function 

$$f(x) = P^2(x) + x^2 - x_0^2.$$ 

All common points of $C$ and $K$ solve $f(x) = 0$ and vice-versa. Now, 

$$f(x_0) = 0, \quad \lim_{x \to \pm \infty} f(x) = \infty,$$

$$f'(x_0) = 2x_0 \neq 0.$$ 

Therefore, $f(x)$ must take a negative value somewhere, because otherwise $x_0$ would be a global minimum, and that would contradict $f'(x_0) \neq 0$. If that happens for $x_1 > x_0$, then we apply the intermediate value theorem for $f(x)$ on $[x_1, A]$, where $A > x_1$ is such that $f(A) > 0$ (such an $A$ exists because $f(x_1) \to \infty$, as $x \to \infty$). If $x_1 < x_0$, then we do the same thing on $[B, x_1]$, where $B_1 < x_1$ and $f(B) > 0$. In either case, we get another zero of $f(x)$.

At least partially solved by:

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