Problem: Let $A > 0$ be a rational number. Show that $A$ is the area of a right triangle with rational sides if and only if there exist three rational numbers $u, v$ and $w$, so that $u^2 - v^2 = v^2 - w^2 = A$.

Solution (by Nguyen Nguyen T.K., San Antonio College; edited by the Panel)

Let $A$ be the area of the right triangle $T$ with rational sides $a, b, c$ ($a \leq b < c$). Then $a^2 + b^2 = c^2$ and $A = \frac{ab}{2}$. Let $u = \frac{a+b}{2}$, $v = \frac{c}{2}$, and $w = \frac{b-a}{2}$. Then $u, v$, and $w$ are rational numbers, and we have

$$A = \frac{ab}{2} = \frac{a^2 + 2ab + b^2}{4} - \frac{a^2 + b^2}{4} = \left(\frac{a+b}{2}\right)^2 - \left(\frac{c}{2}\right)^2 = u^2 - v^2,$$

and

$$A = \frac{ab}{2} = \frac{a^2 + b^2}{4} - \frac{a^2 - 2ab + b^2}{4} = \left(\frac{c}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 = v^2 - w^2.$$

On the other hand, let $u > v > w \geq 0$ be rational numbers such that $u^2 - v^2 = v^2 - w^2 = A$, \hspace{1cm} (*)

for some (rational number) $A$. Let $a = u - w, b = u + w, c = 2v$. Then $a, b, c$ are rational numbers and since $u^2 + w^2 = 2v^2$ by (*), we have:

$$a^2 + b^2 = (u - w)^2 + (u + w)^2 = 2(u^2 + w^2) = 4v^2 = c^2.$$

Since $u^2 - w^2 = 2(v^2 - w^2) = 2A$ by (*), we have:

$$ab = \frac{(u-w) \cdot (u+w)}{2} = \frac{u^2 - w^2}{2} = A.$$

Therefore, $a, b, c$ are rational sides of a right triangle of area $A$.

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