Problem of the Week
Solution of Problem No. 13 (Spring 2007 Series)

Problem: Determine the polynomial \( p(x) \) of degree up to 3 which minimizes

\[
m = \max_{0 \leq x \leq 1} | \cos 4\pi x - p(x) |.
\]

Prove your answer.

Solution (by Pete Kornya, Faculty, Ivy Tech, Bloomington)

If \( p(x) \) is the zero polynomial, then \( m = \max_{0 \leq x \leq 1} | \cos 4\pi x | = 1 \). Now suppose that \( p(x) \) is a polynomial of degree up to 3 such that \( m \leq 1 \). We show that \( p(x) \) must be the zero polynomial. Since \( m \leq 1 \)

\[
p(0), p(0.5), p(1) \geq 0; \quad p(0.25), p(0.75) \leq 0 \quad (1)
\]

Let \( \Delta \) be the forward differencing operator \( \Delta : p(x) \rightarrow p(x + 0.25) - p(x) \). Then, since the degree of \( p(x) \) is at most 3,

\[
0 = \Delta^4 p(0) \\
= p(1) - 4p(0.75) + 6p(0.5) - 4p(0.25) + p(0) \quad (2) \\
= [p(1) - p(0.75)] + 3[p(0.5) - p(0.75)] + 3[p(0.5) - p(0.25)] + [p(0) - p(0.25)]
\]

By (1) and the last line of (2), \( p(1) = p(0.75) = p(0.5) = p(0.25) = p(0) = 0 \). Since the degree of \( p(x) \) is at most 3, and \( p(x) \) has at least 5 zeros, it must be the zero polynomial. Therefore the required polynomial \( p(x) \) is the zero polynomial.

Also solved by:

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