Problem: Find the last two decimal digits of $2007^{2007}$. Computers not allowed. Show your work.

Solution (by Daniel Vacaru, Pitesti, Romania)

We know that $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (Newton’s binomial theorem); from this fact, $2007^{2007} = (2000 + 7)^{2007} = \sum_{k=0}^{2007} \binom{2007}{k} 2000^{2007-k} 7^k$. From this fact, we deduce the last two digits of $2007^{2007}$. These digits are the same with those of $7^{2007}$. But we have

\[
\begin{align*}
7^1 &= 7 \\
7^2 &= 49 \\
7^3 &= 343 = .43 \\
7^4 &= 2401 = .01
\end{align*}
\]

By induction, we have $7^{4k+1} = \ldots.07$, $7^{4k+2} = \ldots.49$, $7^{4k+3} = .43$, and $7^{4k} = .01$ (because $7^4 = 2401$, and from the algorithm for multiplication). We have, $2007 = 2004 + 3 = 4 \cdot 501 + 3$, and, consequently, the last two digits of $2007^{2007}$ are 43.

Also solved by:

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